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On Limiting Directions of Julia Sets of Entire Solutions of Complex Differential Equations

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Abstract: Assume that f is a transcendental entire function. The ray $\arg z=\theta\in[0,2\pi]$ is said to be a limiting direction of the Julia set $\mathcal{J}(f)$ of f if there exists an unbounded sequence $\{z_n\}\subseteq\mathcal{J}(f)$ such that $\lim\limits_{r_n\to\infty}\arg z_n=\theta$. In this paper, we mainly investigate the dynamical properties of Julia sets of entire solutions of the complex differential equations $F(z)f^n(z)+P(z,f)=0$, and $f^n+A(z)P(z,f)=h(z)$, where $P(z,f)$ is a differential polynomial in f and its derivatives, $F(z)$, $A(z)$ and $h(z)$ are entire functions. We demonstrate the existence of close relationships Petrenko's deviations of the coefficients and the measures of limiting directions of entire solutions of the above two equations.

Key words: Julia set; limiting direction; entire function; Petrenko's deviation

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0 Introduction and Main Results

Let $f(z)$: $\mathbb{C}\to\mathbb{C}$ be a transcendental entire function in the complex plane $\mathbb{C}$, and $f^m(z)=f\circ(f^{m-1})(z), m\in\mathbb{N}$ denote the m-th iterate of $f(z)$. The Fatou set $F(f)$ and Julia set $\mathcal{J}(f)$ are defined by $F(f)=\{z\in\mathbb{C}|\{f^m(z)\}_{m=1}^{\infty}$, which is normal at z and $\mathcal{J}(f)=\mathbb{C}\backslash F(f)$ respectively. Clearly, $F(f)$ is open, and $\mathcal{J}(f)$ is closed and non-empty. For a basic understanding of complex dynamics, please refer to Ref.[1].

Suppose that $f(z)$ is a transcendental entire function in $\mathbb{C}$ and $\arg z=\theta$ is a ray from the origin. The ray $\arg z=\theta$, $(\theta\in[0,2\pi])$ is said to be the limiting direction of $\mathcal{J}(f)$ if there exists an unbounded sequence $\{z_n\}\subseteq\mathcal{J}(f)$ such that $\lim\limits_{r_n\to\infty}\arg z_n=\theta$. Define $\Delta(f)=\{\theta\in[0,2\pi)\}$ the ray $\arg z=\theta$ is a limiting direction of $\mathcal{J}(f)$.

It is known that $\Delta(f)$ is closed and measurable, and we use meas$\Delta(E)$ to stand for its linear measure.

The Nevanlinna theory is an important tool in this paper. We use some standard notations such as proximity function $m(r,f)$, counting function of poles $N(r,f)$, and Nevanlinna characteristic function $T(r,f)$. The order $\rho(f)$ and lower order $\mu(f)$ are defined by

$$\rho(f)=\limsup_{r\to\infty}\frac{\log^+\log^+M(r,f)}{\log r},$$

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$$\mu(f)=\liminf_{r\to\infty}\frac{\log^+\log^+ M(r,f)}{\log r},$$

respectively, where $M(r,f)$ denotes the maximum modulus of f on the circle $|z|=r$. And the deficiency of the values a defined by

$$\delta(a,f)=\liminf_{r\to\infty}\frac{m(r,\frac{1}{f-a})}{T(r,f)}.$$

We say that a is a Nevanlinna deficient value of $f(z)$ if $\delta(a,f)>0$. Here, when $a=\infty$, we have

$$\delta(\infty,f)=\liminf_{r\to\infty}\frac{m(r,f)}{T(r,f)}.$$

In addition, for a meromorphic function $f(z)$, we use $S(r,f)$ to denote any quantity satisfying $S(r,f)=o(T(r,f))$ for all r outside a possible exceptional set of finite logarithmic measure.

The Lebesgue linear measure of a set $E\subset[1,\infty)$ is $\text{meas}(E)=\int_E \mathrm{d}t$, and the logarithmic measure of a set $F\subset[1,\infty)$ is $m_1(F)=\int_F\frac{\mathrm{d}t}{t}$. The upper and lower logarithmic densities of $m_1(F)=\int_F\frac{\mathrm{d}t}{t}$ are given by

$$\overline{\log\text{dens}F}:=\limsup_{r\to\infty}\frac{m_1(F\cup[1,r))}{\log r}$$

and

$$\underline{\log\text{dens}F}:=\liminf_{r\to\infty}\frac{m_1(F\cap[1,r))}{\log r},$$

respectively.

Many observations on the radial distribution of Julia sets can be found in Refs. [2-6]. Baker[2] observed that, for a transcendental entire function f , $\mathcal{J}(f)$ cannot be contained in any finite set of straight lines. However, this is not true for transcendental meromorphic functions, for example $\mathcal{J}(\tan z)=R$. Qiao[3] showed that $\text{meas}\Delta(f)=2\pi$ when $\mu(f)<1/2$ and $\text{meas}\Delta(f)\geqslant\pi/\mu(f)$ when $\mu(f)\geqslant 1/2$, where $f(z)$ is a transcendental entire function with finite lower order. Thus, a natural question arises: what can we say about the limit directions of entire functions with infinite lower order?

To answer this question, Huang and Wang[7,8] studied the radial distribution of Julia sets of solutions to complex linear differential equations and obtained the following results.

Theorem 1[7] Let $\{f_1,f_2,\cdots,f_n\}$ be a solution base of

$$f^{(n)}+A(z)f=0 \tag{1}$$

where $A(z)$ is a transcendental entire function with finite order, and denote $E=f_1f_2\cdots f_n$. Then

$$\text{meas}\Delta(E)\geqslant\min\{2\pi,\frac{\pi}{\sigma(A)}\}.$$

Remark 1 Actually, Huang and Wang[7] presented an example to illustrate that $E(z)$ in Theorem 1 may occasionally have infinite lower order. In addition, Huang and Wang[8] directly studied the limiting direction of Julia sets of solutions of a class of higher order linear differential equations, and found that every non-trivial solution is of infinite lower order of these equations.

Theorem 2[8] Let $A_i(z)\,(i=0,1,2,\cdots,n-1)$ be the entire functions of infinite order such that A_0 is transcendental and $m(r,A_i)=o(m(r,A_0))\,(i=1,2,\cdots,n-1)$ as $r\to\infty$. Then every non-trivial solution f of the equation

$$f^{(n)}+A_{n-1}f^{(n-1)}+\cdots+A_0f=0 \tag{2}$$

satisfies $\text{meas}\Delta(f)\geqslant\min\{2\pi,\frac{\pi}{\mu(A_0)}\}$.

Since then, the entire solutions of complex differential equations have attracted much attention; for references, please see Refs.[9-16]. For example, under the assumption of Theorem 2, Zhang *et al*[17] proved that $\text{meas}(\Delta(f)\cap(\Delta(f^{(k)})))\geqslant\min\{2\pi,\pi/\mu(A_0)\}$, where k is a positive integer.

Theorem 3[17] Let $A_i(z)\,(i=0,1,2,\cdots,n-1)$ be the entire functions of finite lower order such that A_0 is transcendental and $m(r,A_i)=o(m(r,A_0))\,(i=1,2,\cdots,n-1)$ as $r\to\infty$. Then every non-trivial solution f of Eq. (2) satisfies

$$\text{meas}(\Delta(f)\cap(\Delta(f^{(k)}))\geqslant\min\{2\pi,\pi/\mu(A_0)\},$$

where k is a positive integer.

To obtain a more precise relationship between $T(r,f)$ and $\log M(r,f)$ of an entire function f, Petrenko introduced the so-called Petrenko's deviation as

$$\beta^-(\infty,f)=\liminf_{r\to\infty}\frac{\log M(r,f)}{T(r,f)},$$

$$\beta^+(\infty,f)=\limsup_{r\to\infty}\frac{\log M(r,f)}{T(r,f)}. \tag{3}$$

If $\beta^-(\infty,f)=\beta^+(\infty,f)$, then there exists a $\nu\in(0,1]$ such that

$$T(r,f)\sim\nu\log M(r,f) \tag{4}$$

as $r\to\infty$ outside an exceptional set. An example $f(z)=\mathrm{e}^z$ satisfies (4) with $\nu=1/\pi$. Heittokangas[11] studied the oscillation of solutions of

$$f''+A(z)f=0, \tag{5}$$

where the coefficient $A(z)$ is associated with Petrenko's deviation. In fact, he obtained the lower bound of the exponent of convergence of zeros of the product of two linearly independent solutions, which depends on Petrenko's deviation of the coefficient $A(z)$. Similar to Ref.[11], let $g(z)$ be entire and set

$$\Xi(g):=\{\theta\in[0,2\pi): \limsup_{r\to\infty}\frac{\log^+|g(re^{i\theta}|)}{\log r}<\infty\} \tag{6}$$

and

$$\xi(g):=\frac{1}{2\pi}\cdot \text{meas}(\Xi(g)).$$

Clearly, $0\leqslant\xi(g)\leqslant 1$.

Define the common limiting directions of the derivatives and primitives of an entire function f by $L(f):=\bigcap_{n\in\mathbb{Z}}\Delta(f^{(n)})$, where $f^{(n)}$ denotes the n-th derivative or the n-th integral primitive of f for $n\geqslant 0$ or $n<0$, respectively. Combining the concept of Petrenko's deviation with the results of limiting directions of Julia set of solutions to complex differential equations, Zhang *et al*[17] proved the lower bound of the set of limiting directions of solutions to Eq. (1) has closed relations with the Petrenko's deviation of the coefficient $A(z)$.

Theorem 4[17] Let $v\in(0,1]$ and A be a transcendental entire function that satisfies (4) as $r\to\infty$ outside a set G with $\log\text{dens}(G)<1$. Then every nontrivial solution f of (2) satisfies

$$\text{meas}(L(f))\geqslant 2\pi v.$$

Moreover, let $f_1,f_2,\cdots,f_n$ be a solution base of Eq. (1), and denote $E=f_1\cdot f_2\cdots f_n$. We have

$$\text{meas}(L(E))\geqslant 2\pi v.$$

Regarding Theorem 1-4 and the knowledge of limiting directions of complex differential equations, we aim to study the lower bound of the set of limiting directions of the following differential equation

$$F(z)f^n(z)+P(z,f)=0, \tag{7}$$

where $F(z)$ is a transcendental entire function and it is associated with Petrenko's deviation,

$$P(z,f)=\sum_{j=1}^{s}\alpha_j(z)f^{n_{0j}}(f')^{n_{1j}}\cdots(f^{(k)})^{n_{kj}}$$

is a differential polynomial in $f(z)$ and its derivatives. The powers $n_{0j},n_{1j},\cdots,n_{kj}$ are non-negative integers and satisfy $\gamma_p=\min_{1\leqslant j\leqslant s}(\sum_{i=0}^{k}n_{ij})\geqslant n$ and the meromorphic fuctions $\alpha_j(z)\,(j=1,2,\cdots,s)$ are small functions of $F(z)$.

Theorem 5 Let $v\in(0,1]$ and $F(z)$ be a transcendental entire function that satisfies (4) as $r\to\infty$ outside a set G with $\log\text{dens}(G)<1$. Suppose that n,k are integers and that $P(z,f)$ is a differential polynomial in f with $\gamma_p\geqslant n$, where all coefficient $\alpha_j\,(j=1,2,\cdots,s)$ are small functions of $F(z)$. Then every non-trivial entire solution $f(z)$ of Eq. (7) satisfies

$$\text{meas}(L(f))\geqslant 2\pi v. \tag{8}$$

We recall the Jackson difference operator

$$D_q f(z)=\frac{f(qz)-f(z)}{qz-z}, z\in\mathbb{C}\backslash\{0\}, q\in\mathbb{C}\backslash\{0,1\}.$$

For $k\in\mathbb{N}\cup\{0\}$, the Jackson k-th difference operator is denoted by

$$D_q^0 f(z):=f(z),\ D_q^k f(z):=D_q(D_q^{k-1}f(z)).$$

Clearly, if f is differentiable,

$$\lim_{q\to 1}D_q^k f(z)=f^{(k)}(z).$$

Thus, a natural question arises: for Eq. (7), if we study the Jackson difference operators of f, does the conclusion $\text{meas}(\bigcap_{k\in\mathbb{N}\cup\{0\}}\Delta(D_q^k f(z)))\geqslant 2\pi v$ hold?

Set $R(f)=\bigcap_{k\in\mathbb{N}\cup\{0\}}\Delta(D_q^k f(z))$, where $q\in(0,+\infty)\backslash\{1\}$ and $D_q^k f(z)$ denotes the k-th Jackson difference operators of $f(z)$. Our result can be stated as follows.

Theorem 6 Let $v\in(0,1]$ and $F(z)$ be a transcendental entire function that satisfies (4) as $r\to\infty$ outside a set G with $\log\text{dens}(G)<1$. Suppose that n,k are integers and that $P(z,f)$ is a differential polynomial in f with $\gamma_p\geqslant n$, where $\alpha_j\,(j=1,2,\cdots,s)$ are small functions of $F(z)$. Then we have

$$\text{meas}R(f)\geqslant 2\pi v \tag{9}$$

for every non-trivial entire solution $f(z)$ of Eq. (7) .

In recent decades, due to the introduction of Nevanlinna theory in complex analysis, the properties of solutions of the Tumura-Clunie differential equation have been studied deeply. The original version of the Tumura-Clunie theory was stated by Tumura[6], and the proof was completed by Clunie[18]. Next, we consider a general class of the Tumura-Clunie type non-linear differential equation

$$fn+A(z)P(z,f)=h(z),(n\geqslant 2), \tag{10}$$

where $A(z)$ and $h(z)$ are entire functions, and $P(z,f)=\sum_{j=1}^{s}\alpha_j(z)f^{n_{0j}}(f')^{n_{1j}}\cdots(f^{(k)})^{n_{kj}}$ is a differential polynomial in $f(z)$ and its derivatives. The powers $n_{0j},n_{1j},\cdots,n_{kj}$ are non-negative integers and satisfy $\gamma_p=\min_{1\leqslant j\leqslant s}(\sum_{i=0}^{k}n_{ij})\geqslant n$ and the meromorphic functions $\alpha_j(z)\,(j=1,2,\cdots,s)$ are small functions of $h(z)$. Indeed, we obtain the following results.

Theorem 7 Let f be a nontrivial solution of Eq. (10), where $A(z)$ is an entire function such that $\xi(A)>0$ and $h(z)$ is an entire function with $\beta^-(\infty,h)\geqslant\dfrac{1}{1-\xi(A)}$. Then

$$\text{meas}(L(f))\geqslant\min\left\{2\pi, 2\pi\left(\frac{1}{\beta^-(\infty,h)}+\xi(A)-1\right)\right\}.$$

For an entire function $f(z)=\sum_{n=0}^{\infty}a_n z^{\lambda_n}$, if $f(z)$ satisfies the gaps condition $\frac{\lambda_n}{n}\to\infty$ as $n\to\infty$, we call $f(z)$ is an entire function with Fabry gaps. It satisfies

$$\log L(r,f)\sim\log M(r,f), L(r,f)=\min_{|z|=r}|f(z)| \quad (11)$$

as $r\to\infty$ outside a set of zero logarithmic density. We know that an entire function f with Fabry gaps satisfies $\beta^-(\infty,f)=1$, this yields the following immediate consequence of Theorem 3.

Theorem 8 Let f be a nontrivial solution of Eq. (10), where $h(z)$ is a transcendental entire function with Fabry gaps. Then $\text{meas}(L(f))\geqslant 2\pi\xi(A)$.

2 Preliminary Lemmas

Before introducing lemmas and completing the proof of Theorems, we recall the Nevanlinna characteristic in an angle, see Refs. [10, 14]. Assuming $0<\alpha<\beta<2\pi$, $k=\pi/(\beta-\alpha)$, we denote

$$\Omega(\alpha,\beta)=\{z\in\mathbb{C}|\arg z\in(\alpha,\beta)\},$$
$$\Omega(\alpha,\beta,r)=\{z\in\mathbb{C}|z\in\Omega(\alpha,\beta),|z|<r\},$$
$$\Omega(r,\alpha,\beta)=\{z\in\mathbb{C}|z\in\Omega(\alpha,\beta),|z|>r\},$$

and use $\bar{\Omega}(\alpha,\beta)$ to denote the closure of $\Omega(\alpha,\beta)$.

Let $f(z)$ be meromorphic on the angular $\Omega(\alpha,\beta)$, we define

$$A_{\alpha,\beta}(r,f)=\frac{k}{\pi}\int_1^r\left(\frac{1}{t^k}-\frac{t^k}{r^{2k}}\right)\left\{\log^+\left|f\left(te^{i\alpha}\right)\right|+\log^+\left|f\left(te^{i\beta}\right)\right|\right\}\frac{dt}{t},$$

$$B_{\alpha,\beta}(r,f)=\frac{2k}{\pi r^k}\int_\alpha^\beta\log^+\left|f\left(re^{i\theta}\right)\right|\sin k(\theta-\alpha)d\theta,$$

$$C_{\alpha,\beta}(r,f)=2\sum_{1<|b_v|<r}\left(\frac{1}{|b_v|^k}-\frac{|b_v|^k}{r^{2k}}\right)\sin k\left(\beta_v-\alpha\right),$$

where $b_v=|b_v|e^{i\beta_v}(v=1,2,\cdots)$ are the poles of $f(z)$ in $\Omega(\alpha,\beta)$, counting multiplicities. The Nevanlinna angular characteristic function is defined by

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f).$$

Especially, we use $\sigma_{\alpha,\beta}(f)=\limsup_{r\to\infty}\frac{\log S_{\alpha,\beta}(r,f)}{\log r}$ to denote the order of $S_{\alpha,\beta}(r,f)$.

Lemma 1[19] If f is a transcendental entire function, then the Fatou set of f has no un-bounded multiply connected component.

Lemma 2[20] Suppose $f(z)$ is analytic in $\Omega(r_0,\theta_1,\theta_2)$, U is a hyperbolic domain and f: $\Omega(r_0,\theta_1,\theta_2)\to U$. If there exists a point $a\in\partial U\backslash\{\infty\}$ such that $C_U(a)>0$, then there exists a constant $d>0$ such that for sufficiently small $\varepsilon>0$, we have

$$|f(z)|=O(|z|^d), z\in\Omega(r_0,\theta_1+\varepsilon,\theta_2-\varepsilon), |z|\to\infty.$$

Remark 2 The open set W is called a hyperbolic domain if $\overline{\mathbb{C}}\backslash W$ has at least two points. For an $a\in\mathbb{C}\backslash W$, we set

$$\mathbb{C}_W(a)=\inf\{\lambda_W(z)|z-a|:\forall z\in W\},$$

where $\lambda_W(z)$ is the hyperbolic density on W. It is well known that if every component of W is simply connected, then $C_W(a)\geqslant\frac{1}{2}$. Before introducing the following lemma, we recall the definition of R-set. Suppose that the set $B(z_n,r_n)=\{z\in\mathbb{C}:|z-z_n|<r_n\}$, if $\sum_{n=1}^{\infty}r_n<\infty, z_n\to\infty$, then we call $\bigcup_{n=1}^{\infty}B(z_n,r_n)$ a R-set. Obviously, $\{|z|: z\in\bigcup_{n=1}^{\infty}B(z_n,r_n)\}$ is a set of the finite linear measure.

Lemma 3[8] Let $z=r\exp(i\psi), r_0+1<r$ and $\alpha\leqslant\psi\leqslant\beta$, where $0<\beta-\alpha\leqslant 2\pi$. Suppose that $n(\geqslant 2)$ is an integer, and that $f(z)$ is analytic in $\Omega(r_0,\alpha,\beta)$ with $\sigma_{\alpha,\beta}<\infty$. Choose $\alpha<\alpha_1<\beta_1<\beta$. Then, for every $\varepsilon\in(0,\frac{\beta_j-\alpha_j}{2})(j=1,2,\cdots,n-1)$ outside a set of linear measure zero with

$$\alpha_j=\alpha+\sum_{s=1}^{j-1}\varepsilon_s \text{ and } \beta_j=\beta+\sum_{s=1}^{j-1}\varepsilon_s, \ (j=2,3,\cdots,n-1),$$

there exist $K>0$ and $M>0$ only depending on f, $\varepsilon_1,\cdots,\varepsilon_{n-1}$ and $\Omega(\alpha_{n-1},\beta_{n-1})$, and not depending on z such that

$$\left|\frac{f'(z)}{f(z)}\right|\leqslant Kr^M(\sin k(\psi-\alpha))^{-2}$$

and

$$\left|\frac{f^{(n)}(z)}{f(z)}\right|\leqslant Kr^M\left(\sin k(\psi-\alpha)\prod_{j=1}^{n-1}\sin k_j\left(\psi-\alpha_j\right)\right)^{-2},$$

for all $z\in\Omega(\alpha_{n-1},\beta_{n-1})$ outside an R-set H, where $k=\pi/(\beta-\alpha)$ and $k_{\varepsilon_j}=\pi/(\beta_j-\alpha_j(j=1,2,\cdots,n-1))$.

Remark 3 Ref. [20] proved that Lemma 3 holds when $n=1$, Wu[21] proved the case of $n=2$ and Huang and Wang[8] proved the case of $n>2$.

Lemma 4[16] Suppose that $f(z)$ is a meromorphic function on $\Omega(\alpha-\varepsilon,\beta+\varepsilon)$ for $\varepsilon>0$ and $0<\alpha<\beta<2\pi$. Then for $r>1$ possibly except a set with finite linear measure.

3 Proof of Theorem 5

For a sufficiently large positive constant M_1, define $D:=\{z\in\mathbb{C}:|F(z)|>|z|^{M_1}\}$ and $H(r):=\{\theta\in[0,2\pi): z=re^{i\theta}\in D\}$. Then there exists some $r_1>0$ such that if $r>r_1$, we have

$$2\pi T(r,F)=\int_{H(r)}\log^+|F(re^{i\theta})|d\theta+\log^+|F(re^{i\theta})|d\theta$$
$$\leqslant \text{meas}(H(r))\log M(r,F)$$
$$+M_1\log r(2\pi-\text{meas}(H(r))). \quad (12)$$

Clearly, Eq. (12) leads to

$$2\pi\leqslant \text{meas}(H(r))\frac{\log M(r,F)}{T(r,F)}+\frac{M_1\log r}{T(r,F)}(2\pi-\text{meas}(H(r))). \quad (13)$$

Since $F(z)$ is transcendental and satisfies Eq. (4) outside G, we have

$$\liminf_{n\to\infty}\text{meas}(H(r))\geqslant 2\pi\nu. \quad (14)$$

Therefore, there exists an infinite sequence $\{r_n\}\subset(r_1,+\infty)\backslash G$ such that

$$\liminf_{n\to\infty}\text{meas}(H(r_n))\geqslant 2\pi\nu. \quad (15)$$

We set $j=1,2,\cdots$,

$$B_n:=\bigcup_{j=n}^{\infty}H(r_j).$$

It can be seen that B_n is monotone decreasing measurable set when $n\to\infty$ and $\text{meas}(B_n)\leqslant 2\pi$. Also, we set

$$\tilde{H}:=\bigcap_{n=1}^{\infty}B_n,$$

then $\tilde{H}$ is independent of r. Therefore, according to the monotone convergence Theorem and Eq. (15), we get

$$\text{meas}(\tilde{H})=\lim_{n\to\infty}\text{meas}(B_n)=\lim_{n\to\infty}\text{meas}(\bigcup_{j=n}^{\infty}H(r_j))\geqslant 2\pi\nu. \quad (16)$$

Suppose that $\text{meas}(L(f))<2\pi\nu$. Then $\text{meas}(\tilde{H}\backslash L(f))>0$. Thus, we can choose a open interval $I=(\alpha,\beta)$ such that

$$I\subset\tilde{H}, I\cap L(f)=\varnothing.$$

For every $\theta\in I$, $\arg z=\theta$ is not a limiting direction of the Julia set of some $f^{(k_\theta)}(z)$, where $k_\theta\in\mathbb{Z}$, only depending on θ. We can choose an angular domain $\Omega(\theta-\zeta_\theta,\theta+\zeta_\theta)$ such that

$$(\theta-\zeta_\theta,\theta+\zeta_\theta)\subset I \text{ and } \Omega(r,\theta-\zeta_\theta,\theta+\zeta_\theta)\cap\mathcal{J}(f^{(k_\theta)}(z))=\varnothing, \quad (17)$$

where ζ_θ is a constant depending on θ. From Lemma 1, there exist a related r and an unbounded Fatou component U of $\mathcal{F}(f^{(k_\theta)}(z)$ such that $\Omega(r,\theta-\zeta_\theta,\theta+\zeta_\theta)\subset U$. Take an unbounded and connected closed section Γ on boundary ∂U such that $\mathbb{C}\backslash\Gamma$ is connected. From Remark 2, $C_{\mathbb{C}\backslash\Gamma}(a)\geqslant 1/2$. Since $f^{(k_\theta)}(z):\Omega(r,\theta-\zeta_\theta,\theta+\zeta_\theta)\to\mathbb{C}\backslash\Gamma$ is analytic, we have that for given sufficiently small $\varepsilon>0$, there is a constant $d_1>0$ such that

$$|f^{(k_\theta)}(z)|=O(|z|^{d_1}) \quad \text{as} \quad |z|\to\infty \quad (18)$$

for $z\in\Omega(r,\theta-\zeta_\theta+\varepsilon,\theta+\zeta_\theta-\varepsilon)$.

Case 1 Let $k_\theta\geqslant 0$. Deriving from integral operation

$$|f^{(k_\theta-1)}(z)|=\int_0^z|f^{(k_\theta)}(\gamma)||d\gamma|+c_{k_\theta}, \quad (19)$$

where c_{k_θ} is a constant, and the integration path is a straight line segment from 0 to z. From this and Eq. (18), we have $|f^{(k_\theta-1)}(z)|=O(|z|^{d_1+1})$ for $z\in\Omega(r,\theta-\zeta_\theta+\varepsilon,\theta+\zeta_\theta-\varepsilon)$. By repeating the above discussion, it can be inferred that

$$|f(z)|=O(|z|^{d_1+k_{\theta_{ij}}}),\ z\in\Omega(r,\theta-\zeta_\theta+\varepsilon,\theta+\zeta_\theta-\varepsilon). \quad (20)$$

Thus, from the definition of Nevanlinna angular characteristic, we have

$$S_{\theta-\zeta_\theta+\varepsilon,\theta+\zeta_\theta-\varepsilon}(r,f)=O(\log r). \quad (21)$$

Case 2 Let $k_{\theta_i}<0$. For any angular $\Omega(\alpha,\beta)$, we get

$$S_{\alpha,\beta}(f^{(k_\theta+1)})\leqslant S_{\alpha,\beta}(r,\frac{f^{(k_\theta+1)}}{f^{(k_\theta)}})+S_{\alpha,\beta}(r,f^{(k_\theta)}). \quad (22)$$

By Lemma 4, we obtain

$$S_{\alpha,\beta}(r,\frac{f^{(k_\theta+1)}}{f^{(k_\theta)}})\leqslant K_1(\log^+S_{\alpha+\epsilon,\beta-\epsilon}(r,f^{(k_\theta)})+\log r+1), \quad (23)$$

where $\epsilon=\frac{\varepsilon}{|k_\theta|}$, K_1 is a positive constant. Combining Eq. (18), Eq. (22) and Eq. (23), we can get

$$S_{\theta_{ij}-\zeta_{\theta_{ij}}+2\varepsilon+\epsilon,\theta_{ij}+\zeta_{\theta_{ij}}-2\varepsilon-\epsilon}(r,f^{(k_\theta+1)})=O(\log r). \quad (24)$$

Similar to the above, repeating the discussion $|k_\theta|$ times, we get

$$S_{\theta-\zeta_\theta+3\varepsilon,\theta+\zeta_\theta-3\varepsilon}(r,f)=O(\log r). \quad (25)$$

This means that whether k_θ is positive or not, we always have

$$S_{\theta_{ij}-\zeta_{\theta_{ij}}+3\varepsilon,\theta+\zeta_\theta-3\varepsilon}(r,f)=O(\log r). \quad (26)$$

Thus, $\sigma_{\theta-\zeta_\theta+3\varepsilon,\theta+\zeta_\theta-3\varepsilon}<\infty$. According to Lemma 3, there exist two constants $K>0$ and $M_2>0$ such that

$$|\frac{f^{(s)}(z)}{f(z)}|\leqslant Kr^{M_2}, s=1,2,\cdots,k, \quad (27)$$

for all $z\in\Omega(r,\theta-\zeta_\theta+3\varepsilon,\theta+\zeta_\theta-3\varepsilon)$ outside a R-set. From (7), we have

$$|z|^{M_1}<|F(z)|=|\frac{P(z,f)}{f^n}| \quad (28)$$

and

$$|F(z)|=\sum_{j=1}^{s}|\alpha_j(z)(\frac{f'}{f})^{n_{1j}}(\frac{f''}{f})^{n_{2j}}\cdots(\frac{f^{(k)}}{f})^{n_{kj}}f^{n_{0j}+n_{1j}+\cdots+n_{kj}-n}|. \quad (29)$$

Since $n_{0j}+n_{1j}+\cdots+n_{kj}-n\geqslant 0$, we get

$$f^{n_{0j}+n_{1j}+\cdots+n_{kj}-n}=O(|z|^{d_1}) \text{ as } |z|\to\infty. \tag{30}$$

Combining Eq. (28), Eq. (29) and Eq. (30), it is found that

$$|z|^{M_1}<|F(z)|=|\frac{P(z,f)}{f^n}|\leqslant Kr^{M_3}. \tag{31}$$

It is impossible since M_1 can be taken sufficiently large and M_3 is a finite positive constant. Therefore,

$$\text{meas}(L(f))\geqslant 2\pi v.$$

4 Proof of Theorem 6

Similar to the Theorem 5, we deduce that meas($\tilde{H}$)$\geqslant 2\pi v$. Conversely, we assume that meas$(R(f))<2\pi v$. So meas$(\tilde{H}\backslash R(f))>0$. We can therefore select finitely many open intervals $I=(\alpha,\beta)$ such that

$$I\subset\tilde{H},\ (\alpha,\beta)\cap R(f)=\varnothing.$$

For every $\theta\in I$, $\arg z=\theta$ is not a limiting direction of the Julia set of $D_q^k f(z)$, where $k\in\mathbb{N}\cup\{0\}$. We can choose an angular domain $\Omega(\theta-\phi_\theta,\theta+\phi_\theta)$ such that

$$(\theta-\phi_\theta,\theta+\phi_\theta)\subset I,$$
$$\Omega(r,\theta-\phi_\theta,\theta+\phi_\theta)\cap\Delta(D_q^k f(z))=\varnothing, \tag{32}$$

where ϕ_{θ_i} is fixed based on θ_i. From Eq. (32) and Lemma 1, there is an unbounded Fatou component U of $\mathcal{F}(\Delta(D_q^k f(z)))$ such that $\Omega(\theta-\phi_\theta,\theta+\phi_\theta)\subset U$. Take an unbounded and connected closed section Γ on boundary ∂U such that $\mathbb{C}\backslash\Gamma$ is connected. From Remark 2, $C_{\mathbb{C}\backslash\Gamma}(a)\geqslant 1/2$. Since $D_q^k f(z)$:$\theta-\zeta_\phi,\theta+\phi_\theta)\to\mathbb{C}\backslash\Gamma$ is analytic, we have that for given sufficiently small enough $\varepsilon>0$, there is a constant $d_2>0$ such that

$$|D_q^k f(z)|=O(|z|^{d_2}),\ z\in\Omega(\alpha^*,\beta^*), \tag{33}$$

where $\alpha^*=\theta-\phi_\theta+\varepsilon$ and $\beta^*=\theta+\phi_\theta-\varepsilon$.

According to the definition of Jackson k-th difference operator, we have

$$|D_q^k f(z)|=\frac{|D_q^{k-1}f(qz)-D_q^{k-1}f(z)|}{|qz-z|}=O(|z|^{d_2}),\ z\in\Omega(\alpha^*,\beta^*). \tag{34}$$

Thus,

$$|D_q^{k-1}f(qz)-D_q^{k-1}f(z)|=O(|z|^{d_2+1}),\ z\in\Omega(\alpha^*,\beta^*). \tag{35}$$

Therefore, there exists a positive constant C such that

$$|D_q^{k-1}f(qz)-D_q^{k-1}f(z)|\leqslant C(|z|^{d_2+1}),\ z\in\Omega(\alpha^*,\beta^*). \tag{36}$$

There are two situations:

Case 1 Let $q\in(0,1)$. If $|z|$ is large enough, choose a positive integer r that satisfies $(\frac{1}{q})^r\leqslant|z|\leqslant(\frac{1}{q})^{r+1}$. In addition, $1\leqslant|q^r z|\leqslant\frac{1}{q}$ has been obtained. So there exists a constant M_4 such that $|D_q^{k-1}f(\frac{z}{q^t})|\leqslant M_4$ where $z\in\{z|1\leqslant|q^r z|\leqslant\frac{1}{q}\}$. From Eq. (36), it can be concluded that

$$\begin{aligned}D_q^{k-1}f(z)&\leqslant|D_q^{k-1}f(z)-D_q^{k-1}f(qz)|\\&+D_q^{k-1}f(qz)-D_q^{k-1}f(q^2z)+\cdots\\&+|D_q^{k-1}f(q^{r-1}z)-D_q^{k-1}f(q^rz)|+|D_q^{k-1}f(q^rz)|\\&\leqslant C(|z|^{d_2+1})+C(|qz|^{d_2+1})+\cdots+C(|q^{r-1}z|^{d_2+1})+M_4\\&\leqslant rC(1+q^{d_2+1}+\cdots+q^{(r-1)(d_2+1)})|z|^{d_2+1}+M_4\\&=O(|z|^{d_2+1})\end{aligned} \tag{37}$$

Thus,

$$|D_q^{k-1}f(z)|=O(|z|^{d_2+1}),\ z\in\Omega(\alpha^*,\beta^*). \tag{38}$$

By repeating the discussion n times, it can be inferred that

$$|f(z)|=O(|z|^{d_2+k-1}),\ z\in\Omega(\alpha^*,\beta^*). \tag{39}$$

Case 2 Let $q\in(1,+\infty)$. If $|z|$ is large enough, choose a positive integer t that satisfies $q^t\leqslant|z|\leqslant q^{t+1}$. In addition, $1\leqslant|\frac{z}{q^t}|\leqslant q$ has been obtained. So there exists a normal number M_5 such that $|D_q^{k-1}f(\frac{z}{q^t})|\leqslant M_5$ where $z\in\{z|1\leqslant|\frac{z}{q^t}|\leqslant q\}$. From (36), it can be concluded that

$$\begin{aligned}D_q^{k-1}f(z)&\leqslant|D_q^{k-1}f(z)-D_q^{k-1}f(\frac{z}{q})|\\&+|D_q^{k-1}f(\frac{z}{q})-D_q^{k-1}f(\frac{z}{q^2})|+\cdots\\&+|D_q^{k-1}f(\frac{z}{q^{t-1}})-D_q^{k-1}f(\frac{z}{q^t})|+|D_q^{k-1}f(\frac{z}{q^t})|\\&\leqslant C(|\frac{z}{q}|^{d_2+1})+C(|\frac{z}{q^2}|^{d_2+1})+\cdots+C(|\frac{z}{q^t}|^{d_2+1})+M_5\\&\leqslant tC(\frac{1}{q^{d_2+1}}+\frac{1}{q^{2(d_2+1)}}+\cdots+\frac{1}{q^{t(d_2+1)}})|z|^{d_2+1}+M_5\end{aligned} \tag{40}$$

Therefore,

$$|D_q^{k-1}f(z)|=O(|z|^{d_2+1}),\ z\in\Omega(\alpha^*,\beta^*). \tag{41}$$

Similar to case 1, we have

$$|f(z)|=O(|z|^{d_2+k-1}),\ z\in\Omega(\alpha^*,\beta^*), \tag{42}$$

which implies that

$$S_{\alpha^*,\beta^*}(r,f)=O(\log r). \tag{43}$$

Similar as Eq. (26) to Eq. (31), we can get a contradiction. Therefore,

$$\text{meas}R(f)\geqslant 2\pi v.$$

5 Proof of Theorem 7

Since $1\leqslant\beta^-(\infty,h)<\frac{1}{1-\xi(A)}$ and $\xi(A)>0$, there exist constants ε and d which satisfy

$$0<\varepsilon<\frac{1}{\beta^{-}(\infty,h)}-(1-\xi(A)),\ \frac{2}{2+\varepsilon}<d<1.$$

Therefore, we get

$$\frac{2(1-d)}{d}<\varepsilon<\frac{1}{\beta^{-}(\infty,h)}-(1-\xi(A)).$$

Define

$$I_d(r):=\{\theta\in[0,2\pi):\log|h(re^{i\theta})|\geqslant(1-d)\log M(r,h)\}.\quad(44)$$

Then,

$$\begin{aligned}2\pi T(r,h)&=\int_{I_d(r)}\log^{+}|h(re^{i\theta})|d\theta+\log^{+}|h(re^{i\theta})|d\theta\\&\leqslant \text{meas}(I_d(r))\log M(r,h)\\&\quad+(2\pi-\text{meas}(I_d(r)))(1-d)\log M(r,h).\end{aligned}\quad(45)$$

Combining the definition of Eq. (3) yields

$$\limsup_{r\to\infty}\text{meas}(I_d(r))\geqslant 2\pi\left(\frac{1}{d\beta^{-}(\infty,h)}-\frac{1-d}{d}\right).\quad(46)$$

For the choice of ε and d, we deduce from Eq. (46) that there exists an infinite sequence $\{r_n\}$ such that

$$\begin{aligned}\text{meas}(I_d(r_n))&\geqslant 2\pi\left(\frac{1}{d\beta^{-}(\infty,h)}-\frac{1-d}{d}\right)-\pi\varepsilon\\&\geqslant\frac{2\pi}{d\beta^{-}(\infty,h)}-2\pi\varepsilon\\&\geqslant\frac{2\pi}{d\beta^{-}(\infty,h)}-2\pi\left(\frac{1}{\beta^{-}(\infty,h)}-(1-\xi(A))\right)\\&\geqslant 2\pi(1-\xi(A)).\end{aligned}\quad(47)$$

Set $D_n:=\bigcup_{n=j}^{\infty}I_d(r_j)$ and $\tilde{I}_d:=\bigcap_{n=1}^{\infty}D_n$. Similar to Section 3, we have

$$\begin{aligned}\text{meas}(\tilde{I}_d)&=\lim_{r\to\infty}\text{meas}(D_n)\\&=\lim_{r\to\infty}\text{meas}\left(\bigcup_{n=j}^{\infty}I_d(r_j)\right)>2\pi(1-\xi(A)).\end{aligned}$$

Thus, we can conclude that

$$\begin{aligned}&\text{meas}(\tilde{I}_d)-2\pi(1-\xi(A))\\&>2\pi\left(\frac{1}{d\beta^{-}(\infty,h)}-\frac{1-d}{d}\right)-\pi\varepsilon-2\pi(1-\xi(A))\\&=2\pi\left[\xi(A)-\frac{1}{d}\left(1-\frac{1}{\beta^{-}(\infty,h)}\right)-\frac{\varepsilon}{2}\right]\\&>2\pi\left[\xi(A)-\frac{2+\varepsilon}{2}\left(1-\frac{1}{\beta^{-}(\infty,h)}\right)-\frac{\varepsilon}{2}\right]\\&=2\pi\left(\frac{1}{\beta^{-}(\infty,h)}-(1-\xi(A))\right)-\pi\varepsilon\left(2-\frac{1}{\beta^{-}(\infty,h)}\right).\end{aligned}\quad(48)$$

Since ε can be taken sufficiently small, we have

$$\text{meas}(\tilde{I}_d)-2\pi(1-\xi(A))\geqslant 2\pi\left(\frac{1}{\beta^{-}(\infty,h)}+\xi(A)-1\right)>0.$$

Suppose that

$$\text{meas}(L(f))<2\pi\left(\frac{1}{\beta^{-}(\infty,h)}+\xi(A)-1\right).$$

Then there exists an interval (α,β) such that

$$(\alpha,\beta)\subset\tilde{I}_d\cap\Xi(A),\ (\alpha,\beta)\subset L(f)=\varnothing.\quad(49)$$

By the similar arguments in Theorem 1, we deduce that for some integer n_θ,

$$|f^{(n_\theta)}(z)|=O(|z|^{d_2})\quad(50)$$

for $z\in\Omega(r_\theta,\theta-\xi_\theta+\varepsilon,\theta+\xi_\theta+\varepsilon)\subset\Omega(r_\theta,\alpha,\beta)$ as $|z|\to\infty$, where d_2 is a positive constant, ε is a sufficiently small positive constant. Following the same discussion in Theorem 1, we have $S_{\alpha^*,\beta^*}(r,f)=O(\log r)$, where

$$\alpha^*=\theta-\xi+\varepsilon,\beta^*=\theta+\xi-\varepsilon$$

for $n_\theta\geqslant 0$, and

$$\alpha^*=\theta-\xi+\varepsilon+\varepsilon',\beta^*=\theta+\xi-\varepsilon-\varepsilon'$$

for $n_\theta<0$.

This implies that $\sigma_{\alpha^*,\beta^*}(r,f)<\infty$. According to Lemma 3, there exist two constants $K>0$ and $M_6>0$ such that

$$\left|\frac{f^{(s)}(z)}{f(z)}\right|\leqslant Kr^{M_6},s=1,2,\cdots,k.\quad(51)$$

From Eq. (18) to Eq. (20), Eq. (44), and Eq. (51), rewrite Eq. (10), for $z\in\Omega(r_\theta,\alpha^*+\varepsilon,\beta^*+\varepsilon)$ outside an R-set, we have

$$\begin{aligned}|h(z)|&\leqslant|f^n(z)|\\&+|A(z)|\left|\sum_{j=1}^{s}\alpha_j(z)\left(\frac{f'}{f}\right)^{n_{1j}}\left(\frac{f''}{f}\right)^{n_{2j}}\cdots\left(\frac{f^{(k)}}{f}\right)^{n_{kj}}f^{n_{0j}+n_{1j}+\cdots+n_{kj}}\right|.\end{aligned}$$

This is impossible since $h(z)$ is a transcendental entire function. Then the assertion follows.

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复微分方程整函数解的Julia集的极限方向

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摘要: 假设f是一个超越整函数。如果存在无界序列 $\arg z=\theta\in[0,2\pi]$ 使得 $\lim\limits_{r_n\to\infty}\arg z_n=\theta$,称射线 $\arg z=\theta\in[0,2\pi]$ 是f的Julia集的极限方向。本文主要研究复微分方程 $F(z)f''(z)+P(z,f)=0$ 和 $f''+A(z)P(z,f)=h(z)$ 整数解的Julia集的动力学性质, 其中 $P(z,f)$ 是关于f及其导数的微分多项式,并且 $F(z)$、$A(z)$ 和 $h(z)$ 是整函数。我们证明了上述两个方程的系数的Petrenko偏差与整数解的极限方向的测度之间存在密切关系。

关键词: Julia集; 极限方向; 整函数; Petrenko偏差