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Unilateral Global Bifurcation and One-Sign Solutions for Kirchhoff Type Problem in $\mathbb{R}^N$

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Abstract: In this paper, we study the following Kirchhoff type problem: $\begin{cases}-M(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x)\,\Delta u=\lambda a(x)f(u), & x\in\mathbb{R}^N,\\ u=0, & \text{as}\,|x|\to+\infty.\end{cases}$ Unilateral global bifurcation result is established for this problem. As applications of the bifurcation result, we determine the intervals of λ for the existence, nonexistence, and exact multiplicity of one-sign solutions for this problem.

Key words: unilateral global bifurcation; one-sign solutions; Kirchhoff type problem

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0 Introduction

Consider the following semi-linear elliptic problem

$$\begin{cases}-\Delta u=\lambda a(x)f(u), & x\in\mathbb{R}^N,\\ u=0, & \text{as}\,|x|\to+\infty,\end{cases}\tag{1}$$

where λ is a real parameter, $N\geqslant 3$, and $a\in C^{\alpha}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R})$ for some $\alpha\in(0,1)$ is a weighted function which can be sign-changing and $f\in C(\mathbb{R},\mathbb{R})$, and $f(s)s>0$ for any $s\neq 0$. Edelson *et al*[1,2] studied the existence of positive solution and the existence of global branches of minimal solutions of the problem (1) by the Schauder-Tychonoff fixed point theorem and Dancer global bifurcation theorems[3], respectively. By using Rabinowitz global bifurcation method[4], Rumbos *et al*[5] showed the existence of positive minimal solution of the problem (1). In 2017, Dai *et al*[6] established a global bifurcation result for problem (1).

On the other hand, Lions[7] studied the following Kirchhoff type problem

$$\begin{cases}-M(\int_{\Omega}|\nabla u|^2\mathrm{d}x)\,\Delta u=\lambda a(x)u+g(x,u,\lambda), & \text{in}\ \Omega,\\ u=0, & \text{on}\ \partial\Omega,\end{cases}\tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. The problem (2) is nonlocal as the appearance of the term $\int_{\Omega}|\nabla u|^2\mathrm{d}x$ which implies that it is not a pointwise identity. By applying the bifurcation techniques, Liang *et al*[8] and Figueiredo *et al*[9] also studied equation (2). Dai *et al*[10] studied the problem (2) by Rabinowitz[4].

Motivated by the above papers, we shall study the following Kirchhoff type problem

$$\begin{cases}-M(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x)\,\Delta u=\lambda a(x)[u+g(u)], & x\in\mathbb{R}^N,\\ u=0, & \text{as}\,|x|\to+\infty,\end{cases}\tag{3}$$

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where λ is a real parameter. By Ref. [6], set $I(\Omega):=\left\{a\in C^{\alpha}_{\rm loc}(\Omega,\mathbb{R}):\left\{x\in\Omega: a(x)>0\right\}\neq\varnothing\right\}$.

For any $u\in C_c^{\infty}(\Omega)$ with $\Omega\subseteq\mathbb{R}^N$, we define $\|u\|_1=(\int_{\Omega}|\nabla u|^2\mathrm{d}x)^{1/2}$. Denote by $D^{1,2}(\Omega)$ the completion of $C_c^{\infty}(\Omega)$ with respect to the norm $\|u\|_1$. Denote by $S(\mathbb{R}^N)$ the set of all measurable real functions defined on $\mathbb{R}^N$. Two functions in $S(\mathbb{R}^N)$ are considered as the same element of $S(\mathbb{R}^N)$ when they are equal almost everywhere. Let $L^2(\mathbb{R}^N;|a|):=\left\{u\in S(\mathbb{R}^N):\int_{\mathbb{R}^N}|a|u^2\mathrm{d}x<+\infty\right\}$.

We assume that a, $g(\cdot)$ and $M(\cdot)$ satisfy the following conditions:

(A1) Let $a\in I(\mathbb{R}^N)$. If there exist two continuous positive radially symmetric functions p and P, where $P\in L^{2q'r}(\mathbb{R}^N)$ (where q and r are given in (A3) and Section 1) such that $0<p\leqslant a(x)\leqslant P(|x|)$, $\forall x\in\mathbb{R}^N$ and $\int_{\mathbb{R}^N}|x|^{2-N}P(|x|)\mathrm{d}x<+\infty$.

Furthermore, if P satisfies the following stronger condition (with $r=|x|$)

$$\int_0^{+\infty}r^{N-1}P(r)\mathrm{d}r<+\infty. \tag{4}$$

(A2) $g\in C(\mathbb{R},\mathbb{R})$ is a Holder continuous function with exponent α and $\lim\limits_{s\to\infty}g(s)/s=0$.

(A3) There exist $c>0$ and $q\in(1,2^*]$ such that $|g(s)|\leqslant c(1+|s|^{q-1})$, where

$$2^*=\begin{cases}\dfrac{2N}{N-2}, & N>2,\\ +\infty, & N\leqslant 2.\end{cases}$$

(A4) $M(t)\in C(\mathbb{R}^+)$ is increasing, $M(0)>0$, $\mathbb{R}^+=[0,+\infty)$.

(A5) There exists $m_1>0$, such that $\lim\limits_{t\to+\infty}M(t)=m_1$.

Furthermore, we shall investigate the existence of one-sign solutions for the following Kirchhoff type problems

$$\begin{cases}-M(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x)\Delta u=\lambda a(x)f(u), & x\in\mathbb{R}^N,\\ u=0, & \text{as } |x|\to+\infty.\end{cases} \tag{5}$$

We assume that a satisfies (A1), and f satisfies the following assumptions:

(H1) $f\in C(\mathbb{R},\mathbb{R})$ is a Holder continuous function with exponent α such that $sf(s)>0$ for any $s\neq 0$.

(H2) $f_0, f_\infty\in(0,\infty)$.

(H3) $f_0\in(0,\infty)$, $f_\infty=\infty$.

(H4) $f_0\in(0,\infty)$, $f_\infty=0$.

(H5) $f_0=\infty$, $f_\infty\in(0,\infty)$.

(H6) $f_0=0$, $f_\infty\in(0,\infty)$.

(H7) $f_0=\infty$, $f_\infty=0$.

(H8) $f_0=0$, $f_\infty=\infty$.

(H9) $f_0=0$, $f_\infty=0$.

(H10) $f_0=\infty$, $f_\infty=\infty$.

where $f_0=\lim\limits_{|s|\to 0}\dfrac{f(s)}{s}$, $f_\infty=\lim\limits_{|s|\to\to\infty}\dfrac{f(s)}{s}$.

Finally, we shall study the exact multiplicity of one-sign solutions for (5) by Implicit Function Theorem, the stability properties and condition (A6).

(A6) $f\in C^{\alpha}(\mathbb{R},\mathbb{R})$ such that $f(s)/s$ is decreasing in $(0,+\infty)$ and is increasing in $(-\infty,0)$.

The rest of this paper is arranged as follows. In Section 1, we give some preliminaries and establish the unilateral global bifurcation result for the problem (3). In Section 2, on the above unilateral global bifurcation result, we prove the existence of one-sign solutions for the Kirchhoff type problem (5). In Section 3, we study the exact multiplicity of one-sign solutions for (5).

1 Preliminaries

Let $E:=H^1(\mathbb{R}^N)$ with the norm $\|u\|=(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x)^{1/2}$. Let $P^+=\left\{u\in E\,|\,u(x)>0,\ x\in\mathbb{R}^N\right\}$ and set $P^-=-P^+$ and $P=P^+\cup P^-$.

Now, from Theorem 1.1 in Ref. [6], we know that the following eigenvalue problem

$$\begin{cases}-\Delta u=\lambda a(x)u, & \text{in } \mathbb{R}^N,\\ u(x)\to 0, & \text{as } |x|\to+\infty,\end{cases} \tag{6}$$

possesses a unique principal eigenvalue λ_1, and λ_1 is simple and isolated.

To prove Theorem 1, by Section 4 of Ref. [6], we first consider the following problem

$$\begin{cases}-\Delta u=h(u), & \text{in } \mathbb{R}^N,\\ u(x)\to 0, & \text{as } |x|\to+\infty.\end{cases} \tag{7}$$

Let us define the operator $T:E\to E$ by

$$u(x)=T[h](x)=\int_{\mathbb{R}^N}\varGamma_N(x-y)h(y)\mathrm{d}y, \tag{8}$$

where $\varGamma_N(x-y)=\dfrac{1}{N(N-2)\omega_N}|x-y|^{2-N}$, ω_N being the volume of the unit ball in $\mathbb{R}^N$.

Then by an argument similar to that of Ref. [1], we can show that u is a one-sign $C^{2+\alpha}$ solution of problem (7) if and only if u is a solution of the operator equation $u(x)=T(h)$. Similar to proposition 1 in Ref. [5], we also can show that $T:E\to E$ is linear completely continuous and (8) is equivalent to (7).

The first main result for (3) is the following unilateral global bifurcation theorem.

Theorem 1 Assume that (A1) - (A5) hold. The pair $(\lambda_1 M(0), 0)$ is a bifurcation point of the problem (3) and there are two distinct unbounded continua D^+ and D^- in $\mathbb{R}\times H^1(\mathbb{R}^N)$ of solutions of the problem (3) emanating from $(\lambda_1 M(0), 0)$. Moreover, we have $D^v\subset((\mathbb{R}\times P^v)\cup\{(\lambda_1 M(0),0)\})$, where $\mu\in\{+,-\}$.

Proof By p.5960-5961 in Ref. [6], it is clear that the problem (3) can be equivalently written as $u=G(\lambda,u)=\lambda\cdot\dfrac{T(au)}{M(0)}+H(\lambda,u)$, where

$$H(\lambda,u)=\frac{\lambda(M(0)-M(\|u\|^2))}{M(0)M(\|u\|^2)}T(au)+\frac{T[\lambda a(x)g(u)]}{M(\|u\|^2)}.$$

From conditions (A1)-(A5) and noting $2<2^*$, we can see that H: $\mathbb{R}\times E\to E$ is completely continuous. Furthermore, it follows that G: $\mathbb{R}\times E\to E$ is completely continuous and $G(\lambda,0)=0$, $\forall\lambda\in\mathbb{R}$.

Next, we show $\lim\limits_{\|u\|\to 0} H(\lambda,u)/\|u\|=0$ at $u=0$ uniformly on bounded λ sets. Without loss of generality, we may assume that $q>2$. Otherwise, we can consider $\tilde q=cq$, $c>1$ such that $\tilde q\in(2,2^*)$. From $q<2^*$, we can see $\dfrac{q'(q-2)}{2^*}<\dfrac{2-q'}{2^*}$. So we can choose a real number $r>1$ such that $\dfrac{q'(q-2)}{2^*}\leqslant\dfrac{1}{r}\leqslant\dfrac{2-q'}{2^*}$.

It follows

$$q'r(q-2)\leqslant 2^*,\ q'r'\leqslant 2^*. \tag{9}$$

By (A2) and (A3), for any $\varepsilon>0$, we can choose positive numbers $\delta=\delta(\varepsilon)$ and $M=M(\delta)$ such that the following relations hold: $\left|g(s)/s\right|\leqslant\varepsilon$, for $|s|\leqslant\delta$. $\left|g(s)/s\right|\leqslant M|s|^{q-2}$, for $|s|>\delta$.

Then we can obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}\left|\frac{a(x)g(u)}{u}\right|^{q'r}\mathrm{d}x\\ &\leqslant\varepsilon\int_{\mathbb{R}^N}(a(x))^{q'r}\mathrm{d}x+M^{q'r}\int_{\mathbb{R}^N}(a(x))^{q'r}|u|^{q'r(q-2)}\mathrm{d}x\\ &\leqslant\varepsilon\int_{\mathbb{R}^N}(P(|x|))^{q'r}\mathrm{d}x\\ &\quad+M^{q'r}\Big(\int_{\mathbb{R}^N}(P(|x|))^{2q'r}\mathrm{d}x\Big)^{\frac12}\Big(\int_{\mathbb{R}^N}|u|^{2q'r(q-2)}\mathrm{d}x\Big)^{\frac12}.\end{aligned}$$

By $P\in L^{2q'r}(\mathbb{R}^N)$ and the continuous embedding of $L^{2q'r}(\mathbb{R}^N)\hookrightarrow L^{q'r}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N}(P(|x|))^{2q'r}\mathrm{d}x<+\infty,\ \int_{\mathbb{R}^N}(P(|x|))^{q'r}\mathrm{d}x<+\infty.$$

Moreover, as $u\to+\infty$, we obtain that

$$\left|\frac{a(x)g(u)}{u}\right|^{q'}\to 0\ \text{ in }\ L^r(\mathbb{R}^N).$$

Let $v=u/\|u\|$, by the boundedness of $v\in E$, $q'r'\leqslant 2^*$ and the continuous embedding of $E\hookrightarrow L^{2^*}(\mathbb{R}^N)$, we have $\int_{\mathbb{R}^N}|v|^{q'r'}\mathrm{d}x<c$, furthermore, we can get

$$\begin{aligned}&\int_{\mathbb{R}^N}\left|\frac{a(x)g(u)}{\|u\|}\right|^{q'}\mathrm{d}x=\int_{\mathbb{R}^N}\left|\frac{a(x)g(u)}{|u|}\right|^{q'}|v|^{q'}\mathrm{d}x\\ &\leqslant\Big(\int_{\mathbb{R}^N}\left|\frac{a(x)g(u)}{|u|}\right|^{q'r}\mathrm{d}x\Big)^{\frac1r}\Big(\int_{\mathbb{R}^N}|v|^{q'r'}\mathrm{d}x\Big)^{\frac{1}{r'}}\to 0.\end{aligned}$$

We obtain

$$\lim_{\|u\|\to 0}a(x)g(u)/\|u\|=0,\ \text{in}\ L^{q'}(\mathbb{R}^N) \tag{10}$$

uniformly $x\in\mathbb{R}^N$.

By (10), we have $\lim\limits_{\|u\|\to 0}H(\lambda,u)/\|u\|=0$ uniformly for $x\in\mathbb{R}^N$ and λ on bounded sets, i.e. $H(\lambda,u)=o(\|u\|)$ at $u=0$ uniformly on bounded λ sets.

Furthermore, applying the similar proof method of Theorem 1.3 in Ref. [6] and the Rabinowitz global bifurcation theorem[4], one can obtain that $(\lambda_1 M(0),0)$ is a bifurcation point of the problem (3) and there exists one unbounded continua D of solutions of the problem (3) emanating from $(\lambda_1 M(0),0)$.

Moreover, by the Dancer unilateral global bifurcation theorem[11], we have that there are two distinct unbounded bifurcation continua D^+ and D^- in $\mathbb{R}\times H^1(\mathbb{R}^N)$ of solutions of the problem (3) emanating from $(\lambda_1 M(0),0)$. Moreover, we have

$$D^v\subset((\mathbb{R}\times P^v)\cup\{(\lambda_1 M(0),0)\}),\ \text{where}\ v\in\{+,-\}.$$

2 One-Sign Solutions for Kirchhoff Type Problem

We first have the following results.

Remark 1 From (H1) and (H2), we can see that there exist two positive constants $0<\rho<\sigma$ such that $\rho\leqslant\dfrac{f(s)}{s}\leqslant\sigma$ for all $s\neq 0$.

By an argument similar to that of Lemma 4.1, 4.2 in Ref. [6], we can obtain Lemma 1 and 2.

Lemma 1 Let (H1) and (H2) hold. By Remark 1, the problem (5) has no one-sign solution for any $\lambda\in(\lambda_1 m_1/\rho,+\infty)$.

Lemma 2 Let (H1) and (H2) hold. By Remark 1, the problem (5) has no positive solution for any $\lambda\in(0,\lambda_1 M(0)/\sigma)$. The main results of this section are the following theorem.

Theorem 2 Let (A1), (A4), (A5), (H1) and (H2) hold. For any $\lambda\in(\min\left\{\dfrac{\lambda_1}{f_\infty}m_1,\dfrac{\lambda_1}{f_0}M(0)\right\}$, $\max\left\{\dfrac{\lambda_1}{f_\infty}m_1,\right.$

$\frac{\lambda_1}{f_0}M(0)\}$), the problem (5) possesses two solutions u_1^+, u_1^- such that $u_1^+>0$ and $u_1^-<0$ in $\mathbb{R}^N$. Therefore, we have

$$\lim_{|x|\to+\infty}|x|^{N-2}u_1^+(x)=c_1>0,\ \lim_{|x|\to+\infty}|x|^{N-2}u_1^-(x)=c_2<0$$

for some constants c_1 and c_2.

Proof Let $\zeta:\mathbb{R}^+\to\mathbb{R}^+$ be a continuous function such that $f(u)=f_0u+\zeta(u)$, with $\lim_{|s|\to0}\frac{\zeta(s)}{s}=0$, $\lim_{|s|\to+\infty}\frac{\zeta(s)}{s}=f_\infty-f_0$, uniformly a.e. in $\mathbb{R}^N$. Equation (5) can be divided in the form

$$\begin{cases}-\Delta u=\dfrac{\lambda f_0a(x)u(x)}{M(0)}+H_1(\lambda,u), & \text{in } \mathbb{R}^N,\\ u(x)\to0, & \text{as } |x|\to+\infty,\end{cases}\tag{11}$$

where

$$H_1(\lambda,u)=\frac{\lambda(M(0)-M(\|u\|^2))}{M(0)M(\|u\|^2)}(f_0a(x)u)+\frac{\lambda a(x)\zeta(u)}{M(\|u\|^2)}.$$

Using the same method to prove (10) with obvious changes, it follows that

$$\lim_{\|u\|\to0}\zeta(u)/\|u\|=0,\ \text{in } L^{q'}(\mathbb{R}^N).$$

Moreover, we have

$$\lim_{\|u\|\to0}H_1(\lambda,u)/\|u\|=0,\ \text{in } L^{q'}(\mathbb{R}^N)$$

uniformly on bounded λ sets.

By Theorem 1, there are two distinct unbounded continua D^+ and D^- in $\mathbb{R}\times H^1(\mathbb{R}^N)$ of solutions of the problem (5) emanating from $(\frac{\lambda_1}{f_0}M(0),0)$, such that

$$D^v\subset((\mathbb{R}\times P^v)\cup\left\{(\frac{\lambda_1}{f_0}M(0),0)\right\}),\ \text{where } \mu\in\{+,-\}.$$

We will show that D^v joins $(\frac{\lambda_1}{f_0}M(0),\ 0)$ to $(\frac{\lambda_1}{f_\infty}m_1,\ +\infty)$. Let

$$(\mu_n,u_n)\in D^v\backslash\left\{(\frac{\lambda_1}{f_0}M(0),0)\right\}\ \text{satisfy } |\mu_n|+|u_n|\to+\infty.$$

By Remark 1 and Lemma 2, one can obtain $\mu_n>0$ for all $n\in\mathbb{N}$. It follows from Lemma 1 that there exists a constant M such that $\mu_n\in(0,M]$ for any $n\in\mathbb{N}$. Therefore, we get $\|u_n\|\to+\infty$.

One can get that D^v joins $(\frac{\lambda_1}{f_0}M(0),0)$ to $(\frac{\lambda_1}{f_\infty}m_1,+\infty)$. Let $\xi:\mathbb{R}^+\to\mathbb{R}^+$ be a continuous function such that $f(u)=f_\infty u+\xi(u)$, with

$$\lim_{s\to+\infty}\frac{\xi(s)}{s}=0,\ \lim_{s\to0^+}\frac{\xi(s)}{s}=f_0-f_\infty\tag{12}$$

uniformly a.e. in $\mathbb{R}^N$. We divide the equation

$$\begin{cases}-\Delta u=\dfrac{\lambda f_\infty a(x)u(x)}{m_1}+H_2(\lambda,u), & \text{in } \mathbb{R}^N,\\ u(x)\to0, & \text{as } |x|\to+\infty,\end{cases}\tag{13}$$

where

$$H_2(\lambda,u)=\frac{\lambda(m_1-M(\|u\|^2))}{m_1M(\|u\|^2)}(f_\infty a(x)u)+\frac{\lambda a(x)\xi(u)}{M(\|u\|^2)}.$$

By (12), for any $\varepsilon>0$, we can choose positive numbers $\delta=\delta(\varepsilon)$ and $M=M(\varepsilon)$ such that for a.e. $x\in\mathbb{R}^N$, the following relations hold:

$$|\xi(s)/s|\leqslant\varepsilon,\ \text{for } |s|>\delta;$$
$$|\xi(s)/s|\leqslant M,\ \text{for } |s|\leqslant\delta.$$

Then we can obtain

$$\int_{\mathbb{R}^N}\left|\frac{a(x)\xi(u)}{u}\right|^{q'r}\mathrm{d}x\leqslant\varepsilon\int_{\mathbb{R}^N}(a(x))^{q'r}\mathrm{d}x+M^{q'r}\int_{\mathbb{R}^N}\left|\frac{a(x)}{u}\right|^{q'r}\mathrm{d}x$$
$$\leqslant\varepsilon\int_{\mathbb{R}^N}(P(|x|))^{q'r}\mathrm{d}x$$
$$+M^{q'r}(\int_{\mathbb{R}^N}(P(|x|))^{2q'r}\mathrm{d}x)^{\frac12}(\int_{\mathbb{R}^N}|u|^{2q'r(q-2)}\mathrm{d}x)^{\frac12}.$$

By $P\in L^{2q'r}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N}(P(|x|))^{2q'r}\mathrm{d}x<+\infty,\ \int_{\mathbb{R}^N}(P(|x|))^{q'r}\mathrm{d}x<+\infty.$$

Moreover, as $u\to+\infty$, we obtain

$$\left|\frac{a(x)\xi(u)}{u}\right|^{q'}\to0\ \text{in } L^r(\mathbb{R}^N).$$

Let $v=u/\|u\|$, by the boundedness of $v\in E$, (9) and the continuous embedding of $E\hookrightarrow L^{2^*}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N}|v|^{q'r'}\mathrm{d}x<c.$$

Furthermore, we can get

$$\int_{\mathbb{R}^N}\left|\frac{a(x)\xi(u)}{\|u\|}\right|^{q'}\mathrm{d}x=\int_{\mathbb{R}^N}\left|\frac{a(x)\xi(u)}{|u|}\right|^{q'}|v|^{q'}\mathrm{d}x$$
$$\leqslant(\int_{\mathbb{R}^N}\left|\frac{a(x)\xi(u)}{|u|}\right|^{q'r}\mathrm{d}x)^{\frac1r}(\int_{\mathbb{R}^N}|v|^{q'r'}\mathrm{d}x)^{\frac{1}{r'}}\to0.$$

It follows from that

$$\lim_{\|u\|\to\infty}a(x)\xi(u)/\|u\|=0,\ \text{in } L^{q'}(\mathbb{R}^N)\tag{14}$$

uniformly $x\in\mathbb{R}^N$. Furthermore, one obtain

$$\lim_{\|u\|\to\infty}H_2(\lambda,u)/\|u\|=0,\ \text{in } L^{q'}(\mathbb{R}^N)$$

uniformly for λ on bounded sets.

By the compactness of T^{-1}, we obtain

$$\begin{cases}-\Delta u=\dfrac{\mu f_\infty a(x)u(x)}{m_1}, \text{in } \mathbb{R}^N,\\ u(x)\to0,\quad \text{as } |x|\to+\infty,\end{cases}\tag{15}$$

where $\mu=\lim_{n\to\infty}\mu_n$, again choosing a subsequence and relabeling if necessary. Thus it is clear that $u\in\dot{D}^v\subseteq D^v$ since D^v is closed in $\mathbb{R}\times E$. Moreover, by (15), $\mu f_\infty=\lambda_1m_1$, so that $\mu=\frac{\lambda_1}{f_\infty}m_1$. Thus D^v joins $(\frac{\lambda_1}{f_0}M(0),0)$ to $(\frac{\lambda_1}{f_\infty}m_1,+\infty)$. Now the existence of u_1^+ and u_1^- is clear.

Similar to the proof of the Theorem 1.3 in Ref. [6], we have

$$\lim_{|x|\to+\infty}|x|^{N-2}u_1^+(x)=c_1>0,\ \lim_{|x|\to+\infty}|x|^{N-2}u_1^-(x)=c_2<0$$

for some constants c_1 and c_2.

Theorem 3 Let (A1), (A4), (A5), (H1) and (H3) hold. If $\lambda_1\in(0,\frac{\lambda_1}{f_0}M(0))$, the problem (5) possesses two solutions u_1^+, u_1^- such that $u_1^+>0$ and $u_1^-<0$ in $\mathbb{R}^N$. Therefore, we have

$$\lim_{|x|\to+\infty}|x|^{N-2}u_1^+(x)=c_1>0,\ \lim_{|x|\to+\infty}|x|^{N-2}u_1^-(x)=c_2<0$$

for some constants c_1 and c_2.

Proof Inspired by the idea of Ref. [12], we define the cut-off function of f as the following

$$f^{[n]}(s)=\begin{cases} ns,\ s\in[-\infty,-2n]\cup[2n,+\infty],\\ \dfrac{2n^2+f(-n)}{n}(s+n)+f(-n),\ s\in(-2n,-n),\\ \dfrac{2n^2-f(n)}{n}(s-n)+f(n),\ s\in(n,2n),\\ f(s),\ s\in[-n,n].\end{cases}$$

We consider the following problem

$$\begin{cases}-M(\int_{\mathbb{R}^N}|\nabla u|^2)\Delta u=\lambda a(x)f^{[n]}(u),\ x\in\mathbb{R}^N,\\ u=0, \qquad\qquad \text{as } |x|\to+\infty.\end{cases}\tag{16}$$

Clearly, we can see that $\lim_{n\to+\infty}f^{[n]}(s)=f(s)$, and $(f^{[n]})_\infty=n$.

The Proposition 4.1 in Ref. [13] implies that there exist two sequence unbounded continua $D^{v[n]}$ of solution set of problem (16) emanating from $(\frac{\lambda_1}{nf_\infty}m_1,\infty)$, such that $D^{v[n]}\subset((\mathbb{R}\times P^v)\cup\left\{(\frac{\lambda_1}{nf_\infty}m_1,\infty)\right\})$, where $v\in\{+,-\}$.

Taking $z^*=(0,\infty)$, we easily obtain that $z^*\in\liminf_{n\to+\infty}D^{v[n]}$ with $\|z^*\|_{\mathbb{R}\times E}=+\infty$. So condition (i) of Theorem 1.2 in Ref.[13] is satisfied with $z^*=(0,\infty)$.

Define a mapping $T:\mathbb{R}\times X\to\mathbb{R}\times X$ such that

$$T(\lambda,u)=\begin{cases}(\lambda,\dfrac{u}{\|u\|^2}), & \text{if } 0\leqslant\|u\|<+\infty,\\ (\lambda,0), & \text{if } \|u\|=+\infty,\\ (\lambda,\infty), & \text{if } \|u\|=0.\end{cases}$$

It is easy to verify that T is a homeomorphism and $\|T(z^*)\|_{\mathbb{R}\times E}=0$. Obviously, $\{T(D^{v[n]})\}$ is a sequence of unbounded connected subsets in E, so (ii) of the Theorem 1.2 in Ref. [13] holds. Since $F(\lambda,0)$ is completely continuous from $\mathbb{R}\times E\to E$, we have $(\bigcup_{n=1}^{+\infty}T(D^{v[n]}))\cup\overline{B}_{\mathbb{R}}$ is pre-compact, and accordingly (iii) of the Theorem 1.2 in Ref. [13] holds. Therefore, by the Theorem 1.2 in Ref. [13], $D^v=\limsup_{n\to+\infty}D^{v[n]}$ is unbounded closed connected of solutions of the problem (5) emanating from $(0,\infty)$, and $D^v\subset((\mathbb{R}\times P^v)\cup\{(0,\infty)\})$ by the Proposition 5.1 in Ref. [13], such that either D^v is unbounded in the direction of λ or meets some point on $\{(\lambda^*,0),\lambda^*\in\mathbb{R}\}$.

From (H1) and (H3), we obtain that there exists a positive constant τ such that $f(s)/s\leqslant\tau$ for any $s>0$. So, Lemma 1 implies D^v is bounded in the direction of λ. Hence, D^v meets $(\lambda^*,0)$ for some $\lambda^*\neq0$. From Theorem 2, we can obtain $\lambda^*=\frac{\lambda_1}{f_0}M(0)$ and $(\frac{\lambda_1}{f_0}M(0),0)\in D^v$, where $v=\{+,-\}$. Now the desired conclusion is obvious.

Theorem 4 Let (A1), (A4), (A5), (H1) and (H4) hold. If $\lambda\in(\frac{\lambda_1}{f_0}M(0),+\infty)$, then the problem (5) possesses two solutions u_1^+, u_1^- such that $u_1^+>0$ and $u_1^-<0$ in $\mathbb{R}^N$. Therefore, we have

$$\lim_{|x|\to+\infty}|x|^{N-2}u_1^+(x)=c_1>0,\ \lim_{|x|\to+\infty}|x|^{N-2}u_1^-(x)=c_2<0$$

for some constants c_1 and c_2.

Proof In view of Theorem 2, there are two distinct unbounded continua D^+ and D^- in $\mathbb{R}\times H^1(\mathbb{R}^N)$ of solutions of the problem (5) emanating from $(\frac{\lambda_1}{f_0}M(0),0)$, such that $D^v\subset((\mathbb{R}\times P^v)\cup\left\{(\frac{\lambda_1}{f_0}M(0),0)\right\})$, where $v\in\{+,-\}$.

We only need to show that D^v joins $(\frac{\lambda_1}{f_0}M(0),0)$ to (∞,∞). We shall only prove the case $v=+$ since the proof for the other case is completely analogous.

Suppose on the contrary that there exists λ_M be a blow-up point and $\lambda_M<+\infty$. Then there exists a sequence (λ_n,u_n) such that $\lim_{n\to\infty}\lambda_n=\lambda_M$ and $\lim_{n\to\infty}u_n=+\infty$. Let $v_n=u_n/\|u_n\|$. Then v_n should be the solutions of problem

$$\begin{cases}-\Delta v_n=\dfrac{\lambda a(x)}{M(\|u_n\|^2)}\cdot\dfrac{f(u_n)}{\|u_n\|},\ \text{in } \mathbb{R}^N,\\ v_n(x)\to0, \qquad\qquad \text{as } |x|\to+\infty.\end{cases}\tag{17}$$

Similar to the proof of (14), we can show

$$\lim_{n\to\infty}\frac{f(u_n)}{\|u_n\|}=0,\ \text{in } L^{q'}(\mathbb{R}^N).$$

By the compactness of $G(\lambda,\cdot)$ and (17), we obtain that for some convenient subsequence $v_n\to v_0\equiv0$. This contradicts $\|v_0\|=1$.

Similar to the proof of Theorem 2, we have

$$\lim_{|x|\to+\infty}|x|^{N-2}u_1^+(x)=c_1>0,\ \lim_{|x|\to+\infty}|x|^{N-2}u_1^-(x)=c_2<0$$

for some constants c_1 and c_2.

Theorem 5 Let (A1), (A4), (A5), (H1) and (H5) hold. If $\lambda \in (0, \frac{\lambda_1}{f_\infty} m_1)$, then the problem (5) possesses two solutions u_1^+, u_1^- such that $u_1^+ > 0$ and $u_1^- < 0$ in $\mathbb{R}^N$. Therefore, we have

$$\lim_{|x|\to+\infty} |x|^{N-2} u_1^+(x) = c_1 > 0, \ \lim_{|x|\to+\infty} |x|^{N-2} u_1^-(x) = c_2 < 0$$

for some constants c_1 and c_2.

Proof If (λ, u) is any nontrivial solution of problem (5), dividing problem (5) by $\|u\|^2$ and setting $v = \frac{u}{\|u\|^2}$ yields

$$\begin{cases} -M(\|u\|^2)\Delta v = \lambda a(x) \dfrac{f(u)}{\|u\|^2}, & \text{in } \mathbb{R}^N, \\ v(x) \to 0, & \text{as } |x| \to +\infty. \end{cases} \tag{18}$$

Define

$$\tilde{f}(v) = \begin{cases} \|v\|^2 f(\dfrac{v}{\|v\|^2}), & \text{if } v \neq 0, \\ 0, & \text{if } v = 0, \end{cases}$$

and

$$\tilde{M}(\|v\|^2) := \begin{cases} M(\dfrac{1}{\|v\|^2}), & \text{if } v \neq 0, \\ m_1, & \text{if } v = 0. \end{cases}$$

Evidently, problem (18) is equivalent to

$$\begin{cases} -\tilde{M}(\|v\|^2)\Delta v = \lambda a(x) \tilde{f}(v), & \text{in } \mathbb{R}^N, \\ v(x) \to 0, & \text{as } |x| \to +\infty. \end{cases} \tag{19}$$

It is obvious that $(\lambda \tilde{M}(0), 0)$ is always the solution of problem (19). By simple computation, we can show that $\tilde{f}_0 = f_\infty \in (0, \infty)$ and $\tilde{f}_\infty = f_0 = \infty$.

Now, applying Theorem 3 and the inversion $v \to \frac{v}{\|v\|} = u$, we achieve the conclusion.

Theorem 6 Let (A1), (A4), (A5), (H1) and (H6) hold. If $\lambda \in (\frac{\lambda_1}{f_\infty} m_1, +\infty)$, then the problem (5) possesses two solutions u_1^+, u_1^- such that $u_1^+ > 0$ and $u_1^- < 0$ in $\mathbb{R}^N$. Therefore, we have

$$\lim_{|x|\to+\infty} |x|^{N-2} u_1^+(x) = c_1 > 0, \ \lim_{|x|\to+\infty} |x|^{N-2} u_1^-(x) = c_2 < 0$$

for some constants c_1 and c_2.

Proof Applying a similar method as the proof of Theorem 5 and the conclusion of Theorem 4, we can easily get the desired conclusion.

Theorem 7 Let (A1), (A4), (A5), (H1) and (H7) hold. If $\lambda \in (0, +\infty)$, then the problem (5) possesses two solutions u_1^+, u_1^- such that $u_1^+ > 0$ and $u_1^- < 0$ in $\mathbb{R}^N$. Therefore, we have

$$\lim_{|x|\to+\infty} |x|^{N-2} u_1^+(x) = c_1 > 0, \ \lim_{|x|\to+\infty} |x|^{N-2} u_1^-(x) = c_2 < 0$$

for some constants c_1 and c_2.

Proof Define

$$f^{[n]}(s) = \begin{cases} ns, & s \in [-\frac{1}{n}, \frac{1}{n}], \\ [f(\frac{2}{n}) - 1](ns - 2) + f(\frac{2}{n}), & s \in (\frac{1}{n}, \frac{2}{n}), \\ -[f(-\frac{2}{n}) + 1](ns + 2) + f(-\frac{2}{n}), & s \in (-\frac{2}{n}, \frac{1}{n}), \\ f(s), & s \in (-\infty, -\frac{2}{n}] \cup [\frac{2}{n}, +\infty). \end{cases}$$

Clearly, we can see that $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, and $(f^{[n]})_0 = n$.

Theorem 4 implies that there exists a sequence of unbounded components $D^{v[n]}$ of solutions to problem (20) emanating from $(\frac{\lambda_1}{f_0 n} M(0), 0)$ and joins to (∞, ∞).

The Lemma 2.5 in Ref. [13] implies that there exists an unbounded component D^v of $\limsup_{n\to+\infty} D^{v[n]}$ such that $(0, 0) \in D^v$ and $(\infty, \infty) \in D^v$ where $v = +, -$.

Theorem 8 Let (A1), (A4), (A5), (H1) and (H8) hold. If $\lambda \in (0, +\infty)$, then the problem (5) possesses two solutions u_1^+, u_1^- such that $u_1^+ > 0$ and $u_1^- < 0$ in $\mathbb{R}^N$. Therefore, we have

$$\lim_{|x|\to+\infty} |x|^{N-2} u_1^+(x) = c_1 > 0, \ \lim_{|x|\to+\infty} |x|^{N-2} u_1^-(x) = c_2 < 0$$

for some constants c_1 and c_2.

Proof Define

$$f^{[n]}(s) = \begin{cases} \frac{1}{n} s, & s \in [-\frac{1}{n}, \frac{1}{n}], \\ [f(\frac{2}{n}) - \frac{1}{n^2}](ns - 2) + f(\frac{2}{n}), & s \in (\frac{1}{n}, \frac{2}{n}), \\ -[f(-\frac{2}{n}) + \frac{1}{n^2}](ns + 2) + f(-\frac{2}{n}), & s \in (-\frac{2}{n}, \frac{1}{n}), \\ f(s), & s \in (-\infty, -\frac{2}{n}] \cup [\frac{2}{n}, +\infty). \end{cases}$$

Clearly, we can see $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, and $(f^{[n]})_0 = \frac{1}{n} (f^{[n]})_\infty = f_\infty$.

Theorem 3 implies that there exists a sequence of unbounded components $D^{v[n]}$ of solutions to problem (21) emanating from $(\frac{\lambda_1}{f_0} M(0) n, 0)$ and joins to $(0, \infty)$.

The Corollary 2.1 in Ref. [13] implies that there exists an unbounded component $D^{v[n]}$ of $\limsup_{n\to+\infty} D^{v[n]}$ such that $(\infty, 0) \in D^v$ and $(0, \infty) \in D^v$, where $v = +, -$.

Theorem 9 Let (A1), (A4), (A5), (H1) and (H9) hold. There exists a $\lambda^+ > 0$, such that $\lambda \in (\lambda^+, +\infty)$, then the problem (5) possesses two solutions u_1^+, u_1^- such that $u_1^+ > 0$ and $u_1^- < 0$ in $\mathbb{R}^N$. Therefore, we have

$$\lim_{|x|\to+\infty}|x|^{N-2}u_1^+(x)=c_1>0,\ \lim_{|x|\to+\infty}|x|^{N-2}u_1^-(x)=c_2<0$$

for some constants c_1 and c_2.

Proof In view of Theorem 8, there are two distinct unbounded continua D^v of solutions of the problem (5) emanating from $(\infty, 0)$. Similar to the proof of Theorem 4, we can obtain that D^v joins $(\infty, 0)$ to (∞, ∞).

Theorem 10 Let (A1), (A4), (A5), (H1) and (H10) hold. There exists a $\lambda^+>0$, such that $\lambda\in(0,\lambda^+)$, then the problem (5) possesses two solutions u_1^+, u_1^- such that $u_1^+>0$ and $u_1^-<0$ in $\mathbb{R}^N$. Therefore, we have

$$\lim_{|x|\to+\infty}|x|^{N-2}u_1^+(x)=c_1>0,\ \lim_{|x|\to+\infty}|x|^{N-2}u_1^-(x)=c_2<0$$

for some constants c_1 and c_2.

Proof In view of Theorem 7, there are two distinct unbounded continua D^v of solutions of the problem (5) emanating from $(0, 0)$. Similar to the proof of Theorem 3, we can obtain that D^v joins $(0, 0)$ to $(0, \infty)$.

3 Exact Multiplicity of One-Sign Solutions for Problem (5)

Refs. [14,15] studied exact multiplicity of solutions for a semi-linear elliptic equation, respectively.

In this section, we study exact multiplicity of one-sign solutions for problem (5). We first study the local structure of the bifurcation branch D^v $(v=+,-)$ near $(\lambda_1 M(0), 0)$, which is obtained in Theorem 1. Let $\Phi(\lambda,u)=u-G(\lambda,u)$ and

$$S=\overline{\{(\lambda,u)\in\mathbb{R}\times E:\ \Phi(\lambda,u)=0,u\neq 0\}}^{\mathbb{R}\times E}$$

For $\lambda\in\mathbb{R}$ and $0<s<+\infty$, define an open neighborhood of $(\lambda_1 M(0), 0)$ in $\mathbb{R}\times E$ as follows.

$$B_s(\lambda_1 M(0),0)=\{(\lambda,u)\in\mathbb{R}\times E:\ \|u\|+|\lambda-\lambda_1 M(0)|<s\}.$$

Let E_0 be a closed subset of E satisfying $E=\operatorname{span}\{\psi_1\}\oplus E_0$, where ψ_1 is an eigenfunction corresponding to $\lambda_1 M(0)$ with $\|\psi_1\|=1$. According to the Hahn-Banach theorem, we have $l\in E^*$ satisfying $l(\psi_1)=1$ and $E_0=\{u\in E: l(u)=0\}$, where E^* denotes the dual space of E. For any $0<\varepsilon<+\infty$ and $0<\eta<1$, define

$$K^+_{\varepsilon,\eta}=\{(\lambda,u)\in\mathbb{R}\times E: |\lambda-\lambda_1 M(0)|<\varepsilon,\ |l(u)|>\eta\|u\|\}.$$

Obviously, $K^+_{\varepsilon,\eta}$ is an open subset of E, $K_{\varepsilon,\eta}=K^+_{\varepsilon,\eta}\cup K^-_{\varepsilon,\eta}$, with $K^-_{\varepsilon,\eta}=-K^+_{\varepsilon,\eta}$ which are disjoint and open in E.

Similar to the Lemma 6.4.1 in Ref. [16], we can show the following lemma.

Lemma 3 Let $\eta\in(0,1)$, there is $\delta_0>0$ such that for each δ: $0<\delta<\delta_0$, it holds that

$$((S\backslash\{(\lambda_1 M(0),0)\})\cap B_\delta(\lambda_1 M(0),0))\subseteq K_{\varepsilon,\eta}.$$

And there exist $s\in\mathbb{R}$ and a unique $y\in E_0$ such that $v=s\psi_1+y$ and $|s|>\eta\|v\|$, for each $\lambda=\lambda_1 M(0)+o(1)$.

Further, $((S\backslash\{(\lambda_1 M(0),0)\})\cap B_\delta(\lambda_1 M(0),0))\subseteq K_{\varepsilon,\eta}$. $\lambda=\lambda_1 M(0)+o(1)$ and $y=o(s)$ as $s\to 0$ for these solutions (λ, v).

Remark 2 From (H2) and (A6), we can see that $f_0\geqslant f(s)/s\geqslant f_\infty>0$ for any $s\neq 0, f(0)=0$ and $f_0>f_\infty$.

Remark 3 From Lemma 3, we can see that $D=D^+\cup D^-$ near $(\lambda_1 M(0), 0)$ is given by a curve $(\lambda(s),u(s))=(\lambda_1 M(0)+o(1), s\psi_1+o(1))$ for s near 0. Moreover, we can distinguish between two portions of this curve by $s\geqslant 0$ and $s\leqslant 0$.

Now, when a, $M(\cdot)$ and f satisfy the conditions (A1), (A4), (A5), (A6), by Dai *et al*[17] and Afrouzi *et al*[18], we give the definition of linearly stable solution for the problem (5) first.

For any $\varphi\in E$ and positive solution u of problem (5), we can calculate that the linearized eigenvalue problem of (5) at the direction φ is

$$\begin{cases}-\Delta\varphi-\dfrac{\lambda}{M(\|u\|^2)}a(x)f'(u)\varphi=\dfrac{\mu}{M(\|u\|^2)}\varphi, & \text{in } \mathbb{R}^N,\\ \varphi(x)\to 0, & \text{as } |x|\to+\infty.\end{cases}\tag{20}$$

Definition 1 Suppose u is a solution of problem (5). The linear stability of u can be determined by the linearized eigenvalue problem (20). If all eigenvalues of problem (20) are positive, then we call u is stable, otherwise we call it unstable.

The Morse index $M(u)$ of u is defined as the number of negative eigenvalues of problem (20). Call u is degenerate if 0 is an eigenvalue of problem (20), otherwise it is non-degenerate.

The main results of this paper are the following:

Theorem 11 Let (A1), (A4), (A5), (A6) and (H2) hold. If $\lambda\in(\frac{\lambda_1}{f_0}M(0),\frac{\lambda_1}{f_\infty}m_1)$, then the problem (5) has exactly two solutions $u_1^+(\lambda,\cdot)$ and $u_1^-(\lambda,\cdot)$ such that $u_1^+>0$ and $u_1^-<0$ in $\mathbb{R}^N$, and has only the trivial solution for any $\lambda\in(0,\frac{\lambda_1}{f_0}M(0)]\cup[\frac{\lambda_1}{f_\infty}m_1,+\infty)$.

The following lemma is stability result for the positive solution.

Lemma 4 Under the assumptions of Theorem 11, then any solution u of problem (5) is stable and non-degenerate, and their Morse index are $M(u)=0$.

Proof Let u be a solution of problem (5), and let (μ_1,φ_1) be the corresponding principal eigenpair of problem (20) with $\varphi_1>0$ in $\mathbb{R}^N$. Notice that u and φ_1 satisfy

$$\begin{cases} -\Delta u = \dfrac{\lambda}{M(\|u\|^2)} a(x)f(u), & \text{in } \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to +\infty, \end{cases} \tag{21}$$

and

$$\begin{cases} -\Delta \varphi_1 - \dfrac{\lambda}{M(\|u\|^2)} a(x)f'(u)\varphi_1 = \dfrac{\lambda}{M(\|u\|^2)} \varphi_1, & \text{in } \mathbb{R}^N, \\ \varphi_1(x) \to 0, & \text{as } |x| \to +\infty. \end{cases} \tag{22}$$

Multiplying the first equation of problem (22) by u and the first equation of problem (21) by φ_1, subtracting and integrating, we obtain

$$\mu_1 \int_{\mathbb{R}^N} \varphi_1 u \mathrm{d}x = \frac{\lambda}{M(\|u\|^2)} \int_{\mathbb{R}^N} a(x)(f(u) - f'(u)u)\mathrm{d}x.$$

By some simple computations, we can show that it follows from (A6) that $f(s) - f'(s)s \geqslant 0$ for any $s \geqslant 0$. Since $u \geqslant 0$ and $\varphi_1 > 0$ in $\mathbb{R}^N$, we have $\mu_1 > 0$ and the positive solution u must be stable. Similarly, we also have:

Lemma 5 Under the assumptions of Lemma 4, any negative solution u of problem (5) is stable, hence, non-degenerate and Morse index $M(u) = 0$.

Proof of Theorem 11 Define $F: \mathbb{R} \times E \to \mathbb{R}$ by

$$F(\lambda, u) = -\Delta u - \frac{\lambda}{M(\|u\|^2)} a(x)f(u).$$

From Lemma 4 and Lemma 5, we know that any one sign solution (λ, u) of problem (5) is stable. Therefore, at any one-sign solution (λ^*, u^*) for the problem (5), we can apply the Implicit Function Theorem to $F(\lambda, u) = 0$, and all the solutions of $F(\lambda, u) = 0$ near (λ^*, u^*) are on a curve $(\lambda, u(\lambda))$ with $|\lambda - \lambda^*| \leqslant \varepsilon$ for some small $\varepsilon > 0$. Furthermore, by virtue of Remark 3, the unbounded continua D^+ and D^- are all curves.

To complete the proof, it suffices to show that $u_1^+(\lambda, \cdot)(u_1^-(\lambda, \cdot))$ is increasing (decreasing) with respect to λ. We only prove the case of $u_1^+(\lambda, \cdot)$. The proof of $u_1^-(\lambda, \cdot)$ can be given similarly. Since $u_1^+(\lambda, \cdot)$ is differentiable with respect to λ (as a consequence of Implicit Function Theorem), taking the derivative of the first equation of problem (21) by λ, one can obtain that

$$-\Delta\left(\frac{\mathrm{d}u_1^+}{\mathrm{d}\lambda}\right) = -\frac{\lambda}{M(\|u_1^+\|^2)} a(x)f'(u_1^+)\frac{\mathrm{d}u_1^+}{\mathrm{d}\lambda} + \frac{1}{M(\|u_1^+\|^2)} a(x)f'(u_1^+). \tag{23}$$

Multiplying the first equation of problem (23) by u and the first equation of problem (21) by $\dfrac{\mathrm{d}u_1^+}{\mathrm{d}\lambda}$, subtracting and integrating, we obtain $\dfrac{1}{M(\|u_1^+\|^2)v} \displaystyle\int_{\mathbb{R}^N} [\lambda a(x)(f'(u_1^+)u_1^+ - f(u_1^+))\frac{\mathrm{d}u_1^+}{\mathrm{d}\lambda} + f(u_1^+)u_1^+]\mathrm{d}x = 0.$

Remark 2 implies $f(s)s > 0$ for any $s \in \mathbb{R}$. So we get $(f'(u_1^+)u_1^+ - f(u_1^+))\dfrac{\mathrm{d}u_1^+}{\mathrm{d}\lambda} \leqslant 0$ by (A1). While (A6) shows that $f'(u_1^+)u_1^+ - f(u_1^+) \leqslant 0$. Therefore, we have $\dfrac{\mathrm{d}u_1^+}{\mathrm{d}\lambda} \geqslant 0$.

Next we only prove the case of the uniqueness of positive solution of problem (5) since the proof of the uniqueness of negative solution of problem (5) is similar.

Suppose on the contrary that there exist two solutions u_{11}^+ and u_{12}^+ corresponding to λ with $u_{11}^+ \in D^+$ of the problem (5) for $\lambda \in (\lambda_1/f_0, +\infty)$. For $\varepsilon > 0$, take $(\lambda - \varepsilon, u_{\lambda-\varepsilon}^+)$, $(\lambda + \varepsilon, u_{\lambda+\varepsilon}^+) \in D^+$, then $u_{\lambda\pm\varepsilon}^+ \to u_{11}^+$ as $\varepsilon \to 0$. By the monotonicity of u_{12}^+ with respect to λ, we get $u_{\lambda-\varepsilon}^+ \leqslant u_{12}^+ \leqslant u_{\lambda+\varepsilon}^+$. Then $u_{11}^+ = u_{12}^+$.

By an argument as the above, we can show that problem (5) with $\lambda = \dfrac{\lambda_1}{f_0} M(0)$ has only the trivial solution. We can show that problem (5) has no one-sign solution for any $\lambda \in (0, \dfrac{\lambda_1}{f_0} M(0))$. Suppose on the contrary that there exists a positive solution u for the problem (5), we multiply the first equation of problem (21) by ϕ_1, and obtain after integrations by

$$\lambda_1 \int_{\mathbb{R}^N} a(x)u\phi_1 \mathrm{d}x = \frac{1}{M(\|u\|^2)} \int_{\mathbb{R}^N} \frac{f(u)}{u} a(x)u\phi_1 \mathrm{d}x < \frac{\lambda f_0}{M(0)} \int_{\mathbb{R}^N} a(x)u\phi_1 \mathrm{d}x,$$

where ϕ_1 is a positive eigenfunction associated to λ_1. It follows that $\lambda > \dfrac{\lambda_1}{f_0} M(0)$, which contradicts $\lambda \in (0, \dfrac{\lambda_1}{f_0} M(0))$. Similar to the above proof, we can obtain that the problem (5) has no positive solution for any $\lambda \in (\dfrac{\lambda_1}{f_\infty} m_1, +\infty)$. Furthermore, we can obtain that the problem (5) has only the trivial solution for any $\lambda \in (0, \dfrac{\lambda_1}{f_0} M(0)] \cup [\dfrac{\lambda_1}{f_\infty} m_1, +\infty)$.

Theorem 12 Let (A1), (A4), (A5), (A6) and (H4) hold. If $\lambda \in (\dfrac{\lambda_1}{f_0} M(0), +\infty)$, then the problem (5) has exactly two solutions $u_1^+(\lambda, \cdot)$ and $u_1^-(\lambda, \cdot)$ for such that $u_1^+ > 0$ and $u_1^- < 0$ in $\mathbb{R}^N$, and has only the trivial solution for any $\lambda \in (0, \dfrac{\lambda_1}{f_0} M(0)]$.

Proof By Theorem 4 and an argument similar to that of Theorem 11, we can prove it.

Theorem 13 Let (A1), (A4), (A5), (A6) and (H5) hold. If $\lambda \in (0, \dfrac{\lambda_1}{f_\infty} m_1)$, then problem (5) has exactly two solutions $u_1^+(\lambda, \cdot)$ and $u_1^-(\lambda, \cdot)$ for such that $u_1^+ > 0$ and $u_1^- < 0$ in $\mathbb{R}^N$, and has only the trivial solution for any $\lambda \in [\dfrac{\lambda_1}{f_\infty} m_1, +\infty)$.

Proof By Theorem 5 and an argument similar to that of Theorem 11, we can obtain it.

Theorem 14 Let (A1), (A4), (A5), (A6) and (H7) hold. If $\lambda \in (0, +\infty)$, then the problem (5) has exactly two solutions $u_1^+(\lambda, \cdot)$ and $u_1^-(\lambda, \cdot)$ for such that $u_1^+ > 0$ and $u_1^- < 0$ in $\mathbb{R}^N$.

Proof By Theorem 7 and an argument similar to that of Theorem 11, we can prove it.

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全空间$\mathbb{R}^N$上Kirchhoff型方程单侧全局分歧和保号解

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摘要: 本文研究了下列Kirchhoff型方程: $\begin{cases} -M(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x)\,\Delta u = \lambda a(x) f(u), & x \in \mathbb{R}^N, \\ u = 0, & \text{as} |x| \to +\infty. \end{cases}$ 建立了方程的单侧全局分歧结果。应用上述分歧结果, 对于属于不同区间的值, 得到了方程保号解的存在性,不存在性及解的确切个数。

关键词: 单侧全局分歧; 保号解; Kirchhoff型方程

□