Wuhan University
Journal of Natural Sciences

2024, Vol.29 No.6, 523-528

Article ID 1007-1202(2024)06-0523-06 DOI https://doi.org/10.1051/wujns/2024296523

Cite this article: WANG Chong. The Weighted Embedded Homology of Super-Hypergraphs[J]. *Wuhan Univ J of Nat Sci*, 2024, 29(6): 523-528.

The Weighted Embedded Homology of Super-Hypergraphs

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Abstract: In this paper, we define the weighted embedded homology of super-hypergraphs, give a quasi-partial order and a pseudo-metric on the set made up of all non-vanishing weights on a finite set, and clarify the relationship between the torsion parts of weighted embedded homology with integer coefficients of super-hypergraphs under certain weights.
Key words: Δ-set; super-hypergraph; weighted embedded homology; pseudo-metric
CLC number: O189.22

0 Introduction

Weighted homology is a generalization of homology theory. In general, one of the main motivations of weighted homology is distinguishing different elements in the data set, which was initially defined on a simplicial complex where each simplex is assigned a weight value[1]. Later, Horak and Jost[2] first developed a general framework for Laplace operators defined in terms of the combinatorial structure of a simplicial complex. Ren *et al*[3] developed the theory of weighted persistent homology of weighted simplicial complex, extended the homology of weighted simplicial complexes to the embedded homology of weighted hypergraphs[4], and proved that weighted persistent homology can tell apart filtrations that ordinary persistent homology does not distinguish[5].

Different from simplicial complexes and hypergraphs, the directed edges in a digraph may have two directions, and there may be directed loops in a digraph. Therefore, in this sense, the topological study of digraphs has more general significance in mathematics. In 2020, Wang *et al*[6] studied the persistent homology of vertex-weighted digraphs. They proved the persistent weighted path homology with coefficients in a filed is independent of the choices of weights.

According to Ref. [7], a simplicial complex can be viewed as a Δ-set, and a hypergraph can be represented as a graded subset of Δ-sets. Moreover, the set of allowed elementary paths on a digraph can be expressed as a graded subset of Δ-sets. Hence, our motivation for this paper is to define the weighted embedded homology of graded sets of Δ-sets which are called as super-hypergraphs[8], study the structure of the torsion part of weighted homology groups with coefficients in $\mathbb{Z}$, and consider the relationship between the torsion parts of weighted embedded homology groups with integer coefficients of super-hypergraphs with different weights. The framework of the paper is as follows. Firstly, we respectively review the definition of embedded homology

Received date: 2023-12-20
Foundation item: Supported by the Science and Technology Project of Hebei Education Department (ZD2022168) and the Research and Innovation Team of Cangzhou Normal University (cxtdl2304)
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on super-hypergraphs in Section 1 and define the weighted embedded homology on super-hypergraphs in Section 2. Secondly, in Section 3, we give a quasi-partial order and a pseudo-metric on the set of all weights on a finite set. Notably, Subsection 3.2 provides the relationship between the torsion parts of weighted embedded homology with integer coefficients of super-hypergraphs under certain weights in Theorem 1 and Theorem 2.

1 Preliminaries

Let R be a communicative ring with a unit. Let $X=\{X_n\}_{n\geqslant 0}$ be a Δ-set with face maps

$$d_i: X_n \to X_{n-1},\ 0\leqslant i\leqslant n,$$
$$(x_0x_1\cdots x_n)\mapsto x_0\cdots\hat{x}_i\cdots x_n,\ x_0x_1\cdots x_n\in X_n, \tag{1}$$

such that

$$d_i d_j = d_j d_{i+1} \text{ for } i\geqslant j. \tag{2}$$

A graded subset of X is a sequence of sets $U=\{U_n\}_{n\geqslant 0}$ such that each U_n is a subset of X_n. By Ref. [8, Definition 2.9], a super-hypergraph is a pair (U,X), where X is a Δ-set and U is a graded subset of X. By Refs. [7-9], the infimum chain complex $\text{Inf}_*(U,X)$

$$\text{Inf}_*(U,X)=R(U_n)\cap\partial_n^{-1}R(U_{n-1}),$$

and the supremum chain complex $\text{Sup}_*(U,X)$

$$\text{Sup}_*(U,X)=R(U_n)+\partial_{n+1}R(U_{n+1}),$$

induce the same homology groups, which is called the n-th embedded homology on super-hypergraphs with coefficient in R. That is,

$$H_n(U,X;R)=H_n(\text{Inf}_*(U,X))\cong H_n(\text{Sup}_*(U,X)).$$

Let X and Y be Δ-sets. A Δ-map $f: X\to Y$ is a sequence of functions $f: X_n\to Y_n, n\geqslant 0$, such that $f\circ d_i = d_i\circ f$ for each $0\leqslant i\leqslant n$. Suppose X, Y are Δ-sets and $f: X\to Y$ is a Δ-map. If U, V are graded subsets of X, Y respectively such that $f(U)\subseteq V$, $f:(U,X)\to(V,Y)$ is called a morphism of super-hypergraphs. Then, by Ref. [8, Proposition 2.11], f induces chain maps

$$f_\#^{\text{Inf}}: \text{Inf}_*(U,X)\to\text{Inf}_*(V,Y),$$
$$f_\#^{\text{Sup}}: \text{Sup}_*(U,X)\to\text{Sup}_*(V,Y).$$

The chain maps $f_\#^{\text{Inf}}$ and $f_\#^{\text{Sup}}$ induce homomorphisms $f_\#^{\text{Inf}}$ and $f_\#^{\text{Sup}}$ of homology groups, respectively, such that the following diagram commutes

$$\begin{array}{ccc} H_n(\text{Inf}_*(U,X)) & \xrightarrow{f_*^{\text{Inf}}} & H_n(\text{Inf}_*(V,Y)) \\ l_*\downarrow\cong & & l'_*\downarrow\cong \\ H_n(\text{Sup}_*(U,X)) & \xrightarrow{f_*^{\text{Sup}}} & H_n(\text{Sup}_*(V,Y)) \end{array}$$

Here l is the canonical inclusion of $\text{Inf}_*(U,X)$ into $\text{Sup}_*(U,X)$, and l' is the canonical inclusion of $\text{Inf}_*(V,Y)$ into $\text{Sup}_*(V,Y)$. Letting $f_*=f_*^{\text{Inf}}$ (or alternatively, letting $f_*=f_*^{\text{Sup}}$), we obtain a homomorphism of the embedded homology groups

$$f_*: H_*(U;R)\to H_*(V;R).$$

The following example gives a homomorphism between embedded homology groups with coefficients in $\mathbb{Z}$.

Example 1 Let $R=\mathbb{Z}$. Consider the following two $\mathbb{Z}$-moudules(homology groups)

$$M=\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3\oplus\mathbb{Z}/9=<x>\oplus<y(2,1)>\oplus<y(3,1)>\oplus<y(3,2)>,$$
$$M'=\mathbb{Z}\oplus\mathbb{Z}/9\oplus\mathbb{Z}/4=<x'>\oplus<y'(3,2)>\oplus<y'(2,2)>,$$

Let

$$f(x)=\lambda x'+\mu y'(3,2)+v y'(2,2),\ \lambda,\mu,v\in\mathbb{Z},$$
$$f(y(2,1))=2y'(2,2),$$
$$f(y(3,1))=3y'(3,2),$$
$$f(y(3,2))=\kappa y'(3,2),\ \kappa=0 \text{ or } 3,$$

then f is a homomorphism between M and M'.

2 Weighted Embedded Homology of Super-Hypergraphs

Let X be a Δ-set. Let $w: X_0\to\mathbb{Z}$ be an integer-valued weight. Let $p_i: X_n\to X_0$ be the projection sending $x_0x_1\cdots x_n$ to the i-th point x_i.

We define the weighted boundary map as:

$$\partial_n^w=\sum_{i=0}^{n}(-1)^i(w\circ p_i)\cdot d_i,$$

where d_i are the face maps as defined in Eq. (1). Then,

$$\partial_n^w: \mathbb{Z}<X_n>\to\mathbb{Z}<X_{n-1}>$$

is a homomorphism of free $\mathbb{Z}$-modules given by

$$\partial_n^w(x_0x)=\sum_{i=0}^{n}(-1)^i w(x_i)x_0\cdots\hat{x}_i\cdots x_n$$

for any $x_0x_1\cdots x_n\in X_n$, and extends linearly over $\mathbb{Z}$. Moreover,

$$\partial_{n-1}^w\circ\partial_n^w=0,$$

for any $n\geqslant 1$. Hence, $\mathbb{Z}$-modules

$$C_n=\mathbb{Z}<X_n>,$$

with a weighted boundary map

$$\partial_n^w: C_n\to C_{n-1}$$

is a chain complex.

Let U be a graded subset of X. Let

$$\text{Inf}_n^w(U,X)=R(U_n)\cap(\partial_n^w)^{-1}R(U_{n-1}), \tag{3}$$
$$\text{Sup}_n^w(U,X)=R(U_n)+\partial_{n+1}^w R(U_{n+1}). \tag{4}$$

Then Eq. (3) and Eq. (4) are both sub-chain complexes of $\{C_n,\partial_n^w\}$, called the weighted infimum chain complex of U in X and the weighted supremum chain complex of

U in X, respectively. By Ref. [9], the canonical inclusion

$$\tau\colon \mathrm{Inf}_n^w(U,X)\to \mathrm{Sup}_n^w(U,X)$$

induces an isomorphism in homology;

$$\tau_*\colon H_n(\mathrm{Inf}_n^w(U,X))\to H_n(\mathrm{Sup}_n^w(U,X)),$$

which is called the weighted embedded homology of (U,X), denoted as $H_n^w(U,X;R)$.

For each $n\geqslant 0$, let

$$W_n^w=\{x\in C_n \mid \exists\lambda\in\mathbb{Z},\lambda\neq 0 \text{ and } \exists y\in C_{n+1} \text{ such that } \lambda x=\partial_{n+1}^w y\},$$

where $\{C_n,\partial_n^w\}_{n\geqslant 0}$ is a chain complex. Then by Ref. [1, Theorem 11.4], for each $n\geqslant 0$, there exists $e_1^w,e_2^w,\cdots,e_{l(w)}^w\in C_n$ linearly independent on $\mathbb{Z}$, and there exists $e'^w_1,e'^w_2,\cdots,e'^w_{l(w)}\in W_{n-1}^w$ linearly-independent on $\mathbb{Z}$, such that;

1) $U_n^w=<e_1^w,e_2^w,\cdots,e_{l(w)}^w>$,

2) $W_{n-1}^w=<e'^w_1,e'^w_2,\cdots,e'^w_{l(w)}>$,

3) $\partial_n^w|_{U_n^w}\colon U_n^w\to W_{n-1}^w$,

4) $\partial_n^w\begin{pmatrix} e_1^w\\ \vdots\\ e_{l(w)}^w\end{pmatrix}=\begin{pmatrix} b_1^w & & \\ & \ddots & \\ & & b_{l(w)}^w\end{pmatrix}\begin{pmatrix} e'^w_1\\ \vdots\\ e'^w_{l(w)}\end{pmatrix}$,

where $b_i^w\geqslant 1$ and $b_1^w|b_2^w|\cdots|b_{l(w)}^w$. Note that $l(w),e_i^w,e'^w_i,b_i^w,1\leqslant i\leqslant l(w)$ are all depending on the weight $w\colon X_0\to\mathbb{Z}$.

The torsion part of the n-homology group can be expressed as

$$\mathrm{Tor}(H_n(\{C_k,\partial_k^w\}_{k\geqslant 0}))=\mathbb{Z}/b_1^w\oplus\cdots\oplus\mathbb{Z}/b_{l(w)}^w. \quad (5)$$

By substituting $\{C_k,\partial_k^w\}_{k\geqslant 0}$ with $\mathrm{Inf}_*^w(U,X)$ or $\mathrm{Sup}_*^w(U,X)$ in Eq. (5), we have the structure of the torsion part of $H_n^w(U,X;R)$.

By Ref. [7], the path homology of digraphs can be seen as the embedded homology on super-hypergraphs. Hence, we take the path homology of digraphs as an example to illustrate that the weighted embedded homology is generally different from the unweighted embedded homology. The readers may refer to Refs. [10-13] for details on path homology of digraphs.

Example 2 As shown in Fig. 1, let G be a cycle with vertex set $V=\{v_0,v_1,v_2\}$ and directed edge set:

$$E=\{v_0v_1,v_1v_2,v_2v_0\}.$$

Let $w\colon V\to\mathbb{Z}$ be a weight on G with $w(v_0)=w_0, w(v_1)=w_1$ and $w(v_2)=w_2$. Then,

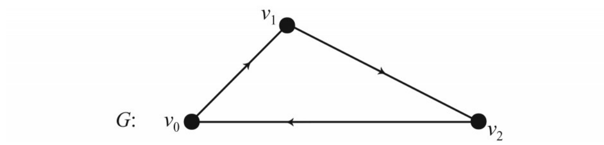

Fig. 1 Example 2

$$\Lambda_0(V)=\mathbb{Z}\{v_0,v_1,v_2\},$$
$$\Lambda_1(V)=\mathbb{Z}\{v_0v_1,v_1v_0,v_1v_2,v_2v_1,v_0v_2,v_2v_0\},$$
$$\Lambda_2(V)=\mathbb{Z}\{v_0v_1v_2,v_0v_1v_0,v_1v_2v_0,v_0v_2v_1,v_0v_2v_0,v_1v_2v_1, v_1v_0v_2,v_1v_0v_1,v_2v_0v_1,v_2v_0v_2,v_2v_1v_2,v_2v_1v_0\},$$
$$\Lambda_3(V)=\mathbb{Z}\{v_0v_1v_2v_0,v_0v_1v_2v_1,v_0v_1v_0v_1,v_0v_1v_0v_2,v_0v_2v_0v_1, v_0v_2v_0v_2,v_0v_2v_1v_0,v_0v_2v_1v_2,v_1v_0v_1v_0,v_1v_0v_1v_2, v_1v_0v_2v_0,v_1v_0v_2v_1,v_1v_2v_0v_1,v_1v_2v_0v_2,v_1v_2v_1v_0, v_1v_2v_1v_2,v_2v_0v_1v_0,v_2v_0v_1v_2,v_2v_0v_2v_0,v_2v_0v_2v_1, v_2v_1v_0v_1,v_2v_1v_0v_2,v_2v_1v_2v_0,v_2v_1v_2v_1\}$$

and

$$\mathcal{A}_0(G)=\mathbb{Z}<v_0,v_1,v_2>,$$
$$\mathcal{A}_1(G)=\mathbb{Z}<v_0v_1,v_1v_2,v_2v_0>,$$
$$\mathcal{A}_2(G)=\mathbb{Z}<v_0v_1v_2,v_1v_2v_0,v_2v_0v_1>,$$
$$\mathcal{A}_3(G)=\mathbb{Z}<v_0v_1v_2v_0,v_1v_2v_0v_1,v_2v_0v_1v_2>,$$

Since,

$$\partial_0^w(v_i)=0, \partial_1^w(v_iv_j)=w(v_i)v_j-w(v_j)v_i=w_iv_j-w_jv_i,$$
$$\partial_2^w(v_iv_jv_k)=w_iv_jv_k-w_jv_iv_k+w_kv_iv_j,$$
$$\partial_3^w(v_iv_jv_kv_l)=w_iv_jv_kv_l-w_jv_iv_kv_l+w_kv_iv_jv_l-w_lv_iv_jv_k,$$

it follows that,

$$\Omega_0^w(G)=\mathbb{Z}<v_0,v_1,v_2>,$$
$$\Omega_1^w(G)=\mathcal{A}_1(G)\cap(\partial_1^w)^{-1}\mathcal{A}_0(G)=\mathbb{Z}<v_0v_1,v_1v_2,v_2v_0>\cap(\partial_1^w)^{-1}\mathbb{Z}<v_0,v_1,v_2>=\mathbb{Z}<v_0v_1,v_1v_2,v_2v_0>$$
$$\Omega_2^w(G)=\mathcal{A}_2(G)\cap(\partial_2^w)^{-1}\mathcal{A}_1(G)=\mathbb{Z}<v_0v_1v_2,v_1v_2v_0,v_2v_0v_1>\cap(\partial_2^w)^{-1}\mathbb{Z}<v_0v_1,v_1v_2,v_2v_0>$$
$$=\begin{cases} 0, & \text{if } w_0,w_1,w_2\neq 0,\\ \mathbb{Z}<v_0v_1v_2>, & \text{if } w_1=0,w_0,w_2\neq 0,\\ \mathbb{Z}<v_1v_2v_0>, & \text{if } w_2=0,w_0,w_1\neq 0,\\ \mathbb{Z}<v_2v_0v_1>, & \text{if } w_0=0,w_1,w_2\neq 0,\\ \mathbb{Z}<v_0v_1v_2,v_1v_2v_0>, & \text{if } w_1=w_2=0,w_0\neq 0,\\ \mathbb{Z}<v_0v_1v_2,v_2v_0v_1>, & \text{if } w_1=w_0=0,w_2\neq 0,\\ \mathbb{Z}<v_1v_2v_0,v_2v_0v_1>, & \text{if } w_0=w_2=0,w_1\neq 0,\\ \mathbb{Z}<v_0v_1v_2,v_1v_2v_0,v_2v_0v_1>, & \text{if } w_0=w_1=w_2=0,\end{cases}$$

and

$$\Omega_3^w(G)=\mathcal{A}_3(G)\cap(\partial_3^w)^{-1}\mathcal{A}_2(G)$$
$$=\mathbb{Z}<v_0v_1v_2v_0,v_1v_2v_0v_1,v_2v_0v_1v_2>\cap(\partial_3^w)^{-1}\mathbb{Z}<v_0v_1v_2,v_1v_2v_0,v_2v_0v_1>$$
$$=\begin{cases}0, & \text{if at least two of } w_0,w_1,w_2\neq 0,\\ \mathbb{Z}<v_0v_1v_2v_0>, & \text{if } w_1=w_2=0, w_0\neq 0,\\ \mathbb{Z}<v_1v_2v_0v_1>, & \text{if } w_2=w_0=0, w_1\neq 0,\\ \mathbb{Z}<v_2v_0v_1v_2>, & \text{if } w_0=w_1=0, w_2\neq 0,\\ \mathbb{Z}<v_0v_1v_2v_0,v_1v_2v_0v_1,v_2v_0v_1v_2>, & \text{if } w_0=w_1=w_2=0.\end{cases}$$

Hence,

$$H_1(G,w;\mathbb{Z})=\mathrm{Ker}\partial_1^w|_{\Omega_1^w(G)}/\,\mathrm{Im}\,\partial_2^w|_{\Omega_2^w(G)}=0, \qquad (6)$$

for $w_1=0$ and $w_0,w_2\neq 0$. By Ref. [10, Proposition 4.7], Eq. (6) is different from the unweighted 1-dimensional path homology group $H_1(G;\mathbb{Z})\cong\mathbb{Z}$.

However, by Refs. [3, 6, 7], the following lemma holds.

Lemma 1 (Ref. [6, Theorem 2.2]) Let $\mathcal{F}$ be any field. Let $C_n(U,X;\mathcal{F})=C_n(U,X;\mathbb{Z})\otimes\mathcal{F}$. Then for any nonvanishing weight w on X,

$$H_*^w(U,X;\mathcal{F})=H_*(U,X;\mathcal{F}).$$

This implies that the weighted homology with $\mathcal{F}$-coefficients does not depend on the choice of w.

Hence, by Lemma 1, we have

Corollary 1 The free part of $H_*^w(U,X;\mathbb{Z})$ does not depend on the choice of non-vanishing weight w.

Proof The free part of $H_*^w(U,X;\mathbb{Z})$ is $\mathbb{Z}^{\oplus r}$, where $r=\dim_{\mathbb{Q}}H_*^w(U,X;\mathbb{Q})$. By Lemma 1, r does not depend on w. Hence, the free part of $H_*^w(U,X;\mathbb{Z})$ does not depend on the choice of w.

3 Main Results

In this section, we give a quasi-partial order and a pseudo-metric on the set made up of all non-vanishing weights on a finite set and consider the morphisms of weighted Δ-sets.

3.1 Vertex Weights on a Finite Set

Let V be a finite set. Let $w,w',w'': V\to\mathbb{Z}\backslash\{0\}$ be non-vanishing weights on V. Using the following notations:

(i) $w=w'$ if for an arbitrary vertex $v\in V$, $w(v)=w'(v)$;

(ii) $w\neq w'$ if there exists a vertex $v\in V$ such that $w(v)\neq w'(v)$;

(iii) $w\approx w'$ if for an arbitrary vertex $v\in V$, $w(v)=\pm w'(v)$;

(iv) $w\not\approx w'$ if there exists a vertex $v\in V$ such that $|w(v)|\neq|w'(v)|$;

(v) $w\lesssim w'$ if for an arbitrary vertex $v\in V$, $w(v)|w'(v)$ ($w(v)$ divides $w'(v)$);

(vi) $w<w'$ if $w\lesssim w'$ and $w\not\approx w'$.

Then

(i) for any weight w on V, $w\approx w$ and $w\lesssim w$ and $w=w$;

(ii) for any weights w and w' on V, if $w\lesssim w'$ and $w'\lesssim w$, then $w\approx w'$;

(iii) for any weights w,w',w'' on V, if $w\lesssim w'$ and $w'\lesssim w''$, then $w\lesssim w''$.

Hence, "$\lesssim$" is a quasi-partial order on the set S_V of all the weights on V. A weight w on V is said minimal, if for any weight w' on V, $w\lesssim w'$. That is,

w is minimal $\leftrightarrow$ $w(v)=\pm 1$ for any vertex $v\in V$.

Two weights w and w' with $w<w'$ are said adjacent if there is no weight w'' on V such that $w<w''<w'$. Hence, $w<w'$ are adjacent if and only if there exists a unique vertex $v\in V$ such that the following both hold:

(i) $w'(v)=\pm pw(v)$ for some prime p;

(ii) $w'(u)=\pm pw(u)$ for any $u\neq v, u\in V$.

Therefore, we have a pseudo-metric d on S_V such that for any two weights w and w' on V ,

(i) $d(w,w')=0$ if $w\approx w'$;

(ii) $d(w,w')=1$ if w and w' are adjacent;

(iii) $d(w,w')=\inf\{|\gamma|\,|\,\gamma=w_0w_1\cdots w_{k-1}w_k$, where $w_0=w$, $w_k=w'$, and w_{i-1} and w_i are adjacent$\}$.

Here $|\gamma|=k$ if $\gamma=w_0w_1\cdots w_{k-1}w_k$.

Note that any w is connected to the minimal weights by certain γ. Hence, $d(w,w')$ make sense for any weights w and w'.

3.2 Morphisms of Super-Hypergraphs

Let X and X' be Δ-sets. Let $w: X_0\to\mathbb{Z}\backslash\{0\}$ and w': $X_0'\to\mathbb{Z}\backslash\{0\}$ be the weights on X and X', respectively. Define a map $f: X_0\to X_0'$ such that for each $x_i\in X_0$,

$$w'(f(x_i))|w(x_i), \qquad (7)$$

and the weight of $(x_0x_1\cdots x_n)\in X_n$ is defined to be the product of weights of all vertices x_i, $0\leqslant i\leqslant n$. Obviously, f is a Δ-map between X and X'. Furthermore, according to Ref. [6, Definition 1.1], since f satisfies Eq. (7), it is referred to as the morphism between weighted Δ-sets. Let U, U' be graded subsets of X, X', respectively. Suppose $f(U) \subseteq U'$, then, f is a weighted morphism of super-hypergraphs.

Lemma 2 (Ref. [6, Theorem 1.1]) A morphism of weighted Δ-sets induces a homomorphism of weighted embedded homology on super-hypergraphs

$$f_*: H_n^w(U,X;\mathbb{Z}) \to H_n^{w'}(U',X';\mathbb{Z}), n \geqslant 0.$$

Notably, in Lemma 2, let $X = X', U = U'$ and f be the identity map τ on X. We have that

Corollary 2 Let w and w' be two weights on X and $w' \lesssim w$. Then, the identity map τ on X induces a homomorphism

$$\tau_*: H_n^w(U,X;\mathbb{Z}) \to H_n^{w'}(U,X;\mathbb{Z}), n \geqslant 0.$$

The following results are based on Corollary 1 and Corollary 2.

Theorem 1 Let w and w' be two weights on X and $w' \lesssim w$. Then, the homomorphism τ_* splits into a direct sum

$$\tau_* = \mathrm{id} \oplus \left(\oplus_{p\ \mathrm{prime}} \mathrm{Tor}_p(\tau_*) \right),$$

where id is the identity map on the free part of $H_n^w(U,X;\mathbb{Z})$ and $\mathrm{Tor}_p(\tau_*)$ is the restriction of τ_* to the p-torsion part

$$\mathrm{Tor}_p(\tau_*): \mathrm{Tor}_p(H_n^w(U,X;\mathbb{Z})) \to \mathrm{Tor}_p(H_n^{w'}(U,X;\mathbb{Z})).$$

Proof We observe that the τ_* sends the p-torsion part of $H_n^w(U,X;\mathbb{Z})$ to the p-torsion part of $H_n^{w'}(U,X;\mathbb{Z})$. Moreover, by Corollary 1 τ_* sends the p-torsion part of $H_n^w(U,X;\mathbb{Z})$ identically to the p-torsion part of $H_n^{w'}(U,X;\mathbb{Z})$. The splitting follows.

Moreover, we have

Theorem 2 Suppose for each $v \in V$, $w(v) = p^\lambda w'(v)$. Here p is a fixed prime and λ is a fixed positive integer. Then, for any prime q,

$$\mathrm{Tor}_q(H_n^w(U,X;\mathbb{Z})) = \begin{cases} p^\lambda \mathrm{Tor}_q(H_n^{w'}(U,X;\mathbb{Z})), & \text{if } q = p, \\ \mathrm{Tor}_q(H_n^{w'}(U,X;\mathbb{Z})), & \text{if } q \neq p. \end{cases}$$

Moreover, $\mathrm{Tor}_q(\tau_*)$ is the identity map if $q \neq p$ and is the canonical projection $\alpha \mapsto p^\lambda \alpha$ if $q = p$.

Proof We observe that

$$\mathrm{Ker}(\partial_n^w) = \mathrm{Ker}(\partial_n^{w'})$$

and

$$\mathrm{Im}(\partial_n^w) = p^\lambda \mathrm{Im}(\partial_n^{w'}).$$

The assertion follows.

Finally, we give an example to illustrate the crucial role of weights in Theorem 2, as shown in Fig. 2. We assign different weights to the vertices of the digraph in Example 2.

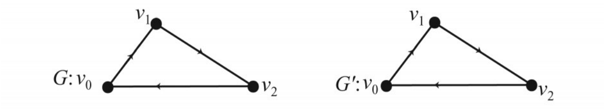

Fig. 2 Example 3

Example 3 Let G be a digraph with the vertex set $\{v_0, v_1, v_2\}$ and the directed edge set

$$\{v_0v_1, v_1v_2, v_2v_0\}.$$

Let w, w' be two weight functions on G such that $w(v_0) = 1, w(v_1) = 2, w(v_2) = 3$ and $w'(v_0) = w'(v_1) = w'(v_2) = 1$.

Consider the two weighted digraphs (G, w) and (G, w'). Let f be the morphism between (G, w) (abbreviated as G) and (G, w') (abbreviated as G') such that $f(v_i) = v_i, 0 \leqslant i \leqslant 2$. By calculation, we have that

$$\partial_1^w(3v_0v_1 + v_1v_2 + 2v_2v_0) = 3(v_1 - 2v_0) + 2v_2 - 3v_1 + 2(3v_0 - v_2) = 0$$

and

$$\partial_1^{w'}(3v_0v_1 + v_1v_2 + 2v_2v_0) = 3(v_1 - v_0) + v_2 - v_1 + 2(v_0 - v_2) = 2v_1 - v_0 - v_2.$$

Hence, $(3v_0v_1 + v_1v_2 + 2v_2v_0) \in \mathrm{Ker}\partial_1^w$ and $(3v_0v_1 + v_1v_2 + 2v_2v_0) \notin \mathrm{Ker}\partial_1^{w'}$.

Remark 1 In Example 3, we have that

$$\mathrm{Ker}(\partial_1^w) \cong \mathrm{Ker}(\partial_1^{w'}),$$
$$\mathrm{Im}\partial_2^w = \mathrm{Im}\partial_2^{w'},$$

and

$$H_1^w(G;\mathbb{Z}) = \mathbb{Z},$$
$$H_1^{w'}(G';\mathbb{Z}) = \mathbb{Z}.$$

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超超图的加权嵌入同调

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摘要：本文定义了超超图的加权嵌入同调，给出了有限集上所有非退化权重做成的集合上的一个拟偏序和一个伪度量，并阐明了在一定权重下超超图整数系数加权嵌入同调群的挠部之间的关系。

关键词：Δ-集；超超图；加权嵌入同调；伪度量

□