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On the Zero Coprime Equivalence of Multivariate Polynomial Matrices

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Abstract: The zero coprime system equivalence is one of important research in the theory of multidimensional system equivalence, and is closely related to zero coprime equivalence of multivariate polynomial matrices. We first discuss the relation between zero coprime equivalence and unimodular equivalence for polynomial matrices. Then, we investigate the zero coprime equivalence problem for several classes of polynomial matrices, some novel findings and criteria on reducing these matrices to their Smith normal forms are obtained. Finally, an example is provided to illustrate the main results.

Key words: multidimensional system; multivariate polynomial matrix; zero coprime equivalence; unimodular equivalence; Smith normal form

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0 Introduction

Multidimensional (nD) systems arise naturally in signal and image processing, linear multi-channel process, iterative learning control system and integrated distributed network synthesis, etc[1-9]. The equivalence of systems is an important research topic in the field of nD systems. It aims at reducing an nD system to an equivalent form with fewer equations and unknowns. Since the behavioral approach to system analysis of an nD system usually resorts to the algebraic property of a multivariate polynomial matrix in the theory of nD systems, the equivalence of nD systems is closely related to the equivalence of nD polynomial matrices. Generally, there are two kinds of the equivalence of nD systems, unimodular system equivalence and zero coprime system equivalence. They correspond to the unimodular equivalence and zero coprime equivalence of nD polynomial matrices, respectively.

For single variable polynomial matrices, the two kinds of equivalence problems have been well resolved since the univariate polynomial ring has the Euclidean division property. However, when it comes to nD ($n \geqslant 2$) polynomial matrices, there are still numerous unresolved issues on the two equivalence problems due to lacking mature theory of nD polynomial matrices. During the past years, the unimodular equivalence for several special classes of nD polynomial matrices have been widely investigated[10-20]. For instance, Lin *et al*[13] proved that a matrix $\boldsymbol{F}(z)$ with $\det(\boldsymbol{F}(z))=z_1-f(z_2,\cdots,z_n)$ is unimodular equivalent to its Smith normal form.

Li *et al*[14-16] and Lu and Zheng *et al*[17-20] also pre-

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sented further results on the unimodular equivalence of several classes of matrices $\boldsymbol{F}(z)$ with $\det(\boldsymbol{F}(z))=(z_1-f(z_2,\cdots,z_n))^t$ or $\det(\boldsymbol{F}(z_1,z_2))=pq$, where t is positive integer and $p,q\in K[z_1]$ are irreducible polynomials, and obtained the sufficient and necessary conditions respectively for the unimodular equivalence of these matrices with their Smith normal forms. Compared with the unimodular equivalence problem, the zero coprime equivalence of nD polynomial matrices has relatively little attention.

Zerz[21] proposed that the stability, controllability and observability of a system are closely related to its basic zero structure. Pugh[22] proved that zero coprime equivalence preserves the zero structure of the system matrix. Furthermore, Pugh *et al*[23] showed that a given bivariate polynomial matrix $\boldsymbol{F}(z_1,z_2)$ is zero coprime equivalent to its first-level and second-level matrix pencil. In addition, Boudellioua[24] proved that nD polynomial matrice $\boldsymbol{F}(z)$ is zero coprime equivalent to the greatest common divisor of the highest order minors of $\boldsymbol{F}(z)$ under given conditions. Although the aforementioned conclusions simplify the corresponding system to a single equation form containing a single unknown, they are not easy to be executed. The Smith normal form plays an important role in the discussion of equivalence of nD systems because of its favorable structure and properties. The main aim of this research is to transform a given nD polynomial matrix into its Smith normal form, by means of zero coprime equivalence, thereby enabling the preservation of important algebraic properties of the corresponding system.

This paper focuses on the zero coprime equivalence problem for several classes of nD polynomial matrices and their Smith normal forms. Based on previous findings of zero prime factorization of nD polynomial matrices[25,26], some new properties on the zero coprime equivalence that are nD polynomial matrices are derived. Firstly, the relation between zero coprime equivalence and unimodular equivalence is discussed. Note that the nD polynomial matrices of unimodular equivalent must be zero coprime equivalent, but the converse is not true. It is natural to associate the zero coprime equivalence problem for several classes of nD polynomial matrices which are not unimodular equivalent to their Smith normal forms. So far, the matrices such as $\boldsymbol{F}(z)$ with $\det(\boldsymbol{F}(z))=(z_1-f_1(z_2,\cdots,z_n))(z_1-f_2(z_2,\cdots,z_n))$ have not been shown to be equivalent to their Smith normal forms. Therefore, the following problems are also investigated.

Problem 1: When is an nD polynomial matrix $\boldsymbol{F}(z)$ zero coprime equivalent to its Smith normal form? Here

$$\boldsymbol{F}(z)=\begin{pmatrix} 1 & f_{12} & \cdots & f_{1,l-1} & f_{1l} \\ 0 & 1 & \cdots & f_{2,l-1} & f_{2l} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & z_1-f_1 & f_{l-1,l} \\ 0 & 0 & \cdots & 0 & z_1-f_2 \end{pmatrix}$$

and $f_1,f_2\in K[z_2,\cdots,z_n]$, $f_{ij}\in K[z_1,\cdots,z_n]$, $1\leqslant i<j\leqslant l$.

Problem 2: When is an nD polynomial matrix $\boldsymbol{F}(z)$ zero coprime equivalent to its Smith normal form? Here

$$\boldsymbol{F}(z)=\begin{pmatrix} z_1-f_1 & f_{12} & \cdots & f_{1,l-1} & f_{1l} \\ 0 & z_1-f_2 & \cdots & f_{2,l-1} & f_{2l} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & z_1-f_{l-1} & f_{l-1,l} \\ 0 & 0 & \cdots & 0 & z_1-f_l \end{pmatrix}$$

and $f_1,f_2,\cdots,f_l\in K[z_2,\cdots,z_n]$, $f_{ij}\in K[z_1,\cdots,z_n]$, $1\leqslant i<j\leqslant l$.

The rest of the paper is organized as follows. In Section 1, some basic concepts for the equivalence of polynomial matrices are introduced. In Section 2, the main results of this paper and positive answers to Problems 1 and 2 are presented. In Section 3, an example is provided to illustrate the main results and the constructive method. Section 4 concludes this paper.

1 Preliminaries

Let K is an algebraic closed field, $R=K[z_1,z_2,\cdots,z_n]$ denotes the set of polynomials in n variables $z_1,z_2,\cdots,z_n$ with coefficients in the field K, sometimes, we denote $K[z_1,z_2,\cdots,z_n]$ by $K[z]$. $R_1=K[z_2,\cdots,z_n]$, $R^{l\times m}$ denotes the set of $l\times m$ matrices with entries from R. $\boldsymbol{0}_{r\times t}$ denotes the $r\times t$ zero matrix and $\boldsymbol{I}_r$ denotes the $r\times r$ identity matrix. Throughout the paper, the argument (z) is omitted whenever its omission does not cause confusion.

Definition 1 Let $\boldsymbol{F}(z)\in R^{l\times m}$ be of normal rank r. For any $i\,(1\leqslant i\leqslant r)$, denote $i\times i$ minors of $\boldsymbol{F}(z)$ by $a_{i1},a_{i2},\cdots a_{i\beta_i}$, and denote the greatest common divisor (g.c.d.) of $a_{i1},a_{i2},\cdots a_{i\beta_i}$ by $d_i(\boldsymbol{F})$.

Definition 2 Let $\boldsymbol{F}(z)\in R^{l\times m}$, $l\leqslant m$, the Smith normal form of $\boldsymbol{F}(z)$ is defined by $\boldsymbol{S}(z)=\begin{pmatrix}\operatorname{diag}\{\varPhi_i\} & \boldsymbol{0}_{l\times(m-l)}\end{pmatrix}$, where

$$\Phi_i=\begin{cases} d_i/d_{i-1}, 1\leqslant i\leqslant r,\\ 0, \qquad r<i\leqslant l,\end{cases}$$

r is the normal rank of $\boldsymbol{F}(z)$, $d_0\equiv 1$, d_i is the g.c.d. of the $i\times i$ minors of $\boldsymbol{F}(z)$ and Φ_i satisfies the following property

$$\Phi_1\big|\Phi_2\big|\cdots\big|\Phi_r.$$

Definition 3 Let $\boldsymbol{F}(z)\in R^{l\times m}$ be of full row(column) rank. $\boldsymbol{F}(z)$ is said to be zero left prime (zero right prime) if all the $l\times l$ ($m\times m$) minors of $\boldsymbol{F}(z)$ generate unit ideal R. If $\boldsymbol{F}(z)$ is zero left prime (zero right prime), then $\boldsymbol{F}(z)$ is called simply to be ZLP (ZRP).

Definition 4 Let $\boldsymbol{A}(z)\in R^{l\times q}, \boldsymbol{B}(z)\in R^{l\times m}, q+m\geqslant l\geqslant 1$. $\boldsymbol{A}(z)$, $\boldsymbol{B}(z)$ are said to be zero left coprime if all the $l\times l$ minors of matrix $(\boldsymbol{A} \quad \boldsymbol{B})$ generate unit ideal R. Zero right coprime can be similarly defined. If $\boldsymbol{A}(z)$, $\boldsymbol{B}(z)$ are zero left coprime (zero right coprime), then $\boldsymbol{A}(z)$, $\boldsymbol{B}(z)$ are called simply to be ZLC (ZRC).

Definition 5 Let $\boldsymbol{P}_i(z)\in R^{l\times m}$, $i=1,2$, then $\boldsymbol{P}_1(z)$, $\boldsymbol{P}_2(z)$ are said to be unimodular equivalent if there exist $\boldsymbol{M}(z)$, $\boldsymbol{N}(z)$ such that $\boldsymbol{M}(z)\boldsymbol{P}_1(z)=\boldsymbol{P}_2(z)\boldsymbol{N}(z)$, where $\boldsymbol{M}(z)$ and $\boldsymbol{N}(z)$ are unimodular matrices over R of appropriate dimensions.

Definition 6[8] Let $\boldsymbol{P}_i(z)\in R^{p_i\times q_i}$, where $i=1,2$ and $p_1-q_1=p_2-q_2$. $\boldsymbol{P}_1(z)$, $\boldsymbol{P}_2(z)$ be related by an equation of the form $\boldsymbol{M}(z)\boldsymbol{P}_2(z)=\boldsymbol{P}_1(z)\boldsymbol{N}(z)$, then $\boldsymbol{P}_1(z)$ and $\boldsymbol{P}_2(z)$ are said to be zero coprime equivalent if $\boldsymbol{M}(z)$, $\boldsymbol{P}_1(z)$ are ZLC and $\boldsymbol{N}(z)$, $\boldsymbol{P}_2(z)$ are ZRC.

2 Main Results

In this section, the main results are presented. First, we give some criteria for the conversion of zero coprime equivalence into unimodular equivalence through trivial expansion. Then we provide some positive answers to Problems 1 and 2.

We first introduce a useful lemma.

Lemma 1[8] The two polynomial matrices $\boldsymbol{A}(z)\in R^{m\times p}$ and $\boldsymbol{B}(z)\in R^{m\times q}$ with $p+q\geqslant m\geqslant 1$ are zero left coprime if and only if there exist $p\times m$ and $q\times m$ polynomial matrices $\boldsymbol{X}(z)$ and $\boldsymbol{Y}(z)$ such that $\boldsymbol{A}(z)\boldsymbol{X}(z)+\boldsymbol{B}(z)\boldsymbol{Y}(z)=\boldsymbol{I}_m$; The two polynomial matrices $\boldsymbol{C}(z)\in R^{p\times m}$ and $\boldsymbol{D}(z)\in R^{q\times m}$ with $p+q\geqslant m\geqslant 1$ are zero right coprime if and only if there exist $m\times p$ and $m\times q$ polynomial matrices $\boldsymbol{W}(z)$ and $\boldsymbol{Z}(z)$ such that $\boldsymbol{W}(z)\boldsymbol{C}(z)+\boldsymbol{Z}(z)\boldsymbol{D}(z)=\boldsymbol{I}_m$.

Theorem 1 Let $\boldsymbol{F}_i(z)\in R^{p_i\times q_i}$, $i=1,2$ be nD polynomial matrices and $p_1-q_1=p_2-q_2$, then $\boldsymbol{F}_1(z)$, $\boldsymbol{F}_2(z)$ are zero coprime equivalent if and only if certain trivial expansions of them, $\begin{pmatrix}\boldsymbol{I}_{q_2} & 0\\ 0 & \boldsymbol{F}_1(z)\end{pmatrix}$ and $\begin{pmatrix}\boldsymbol{I}_{q_1} & 0\\ 0 & \boldsymbol{F}_2(z)\end{pmatrix}$ are unimodular equivalent.

Proof Necessity: Suppose $\boldsymbol{F}_1(z)$, $\boldsymbol{F}_2(z)$ are zero coprime equivalent, then there exist two matrices $\boldsymbol{M}(z)\in R^{p_2\times p_1}$, $\boldsymbol{N}(z)\in R^{q_2\times q_1}$ which satisfy equation $\boldsymbol{M}(z)\boldsymbol{F}_1(z)=\boldsymbol{F}_2(z)\boldsymbol{N}(z)$ and $\boldsymbol{M}(z)$, $\boldsymbol{F}_2(z)$ are ZLC, $\boldsymbol{N}(z)$, $\boldsymbol{F}_1(z)$ are ZRC. According to Lemma 1, we have

$$\boldsymbol{M}(z)\boldsymbol{X}_1(z)+\boldsymbol{F}_2(z)\boldsymbol{Y}_1(z)=\boldsymbol{I}_{p_2}, \boldsymbol{X}_2(z)\boldsymbol{N}(z)+\boldsymbol{Y}_2(z)\boldsymbol{F}_1(z)=\boldsymbol{I}_{q_1},$$

where $\boldsymbol{X}_1(z)$, $\boldsymbol{X}_2(z)$, $\boldsymbol{Y}_1(z)$, $\boldsymbol{Y}_2(z)$ are of appropriate dimensions.

Furthermore, we have

$$\begin{pmatrix}\boldsymbol{X}_2(z) & -\boldsymbol{Y}_2(z)\\ \boldsymbol{F}_2(z) & \boldsymbol{M}(z)\end{pmatrix}\begin{pmatrix}\boldsymbol{N}(z) & \boldsymbol{Y}_1(z)\\ -\boldsymbol{F}_1(z) & \boldsymbol{X}_1(z)\end{pmatrix}=\begin{pmatrix}\boldsymbol{I}_{q_1} & \boldsymbol{J}(z)\\ 0 & \boldsymbol{I}_{p_2}\end{pmatrix},$$

where $\boldsymbol{J}(z)=\boldsymbol{X}_2(z)\boldsymbol{Y}_1(z)-\boldsymbol{Y}_2(z)\boldsymbol{X}_1(z)$.

Let $\overline{\boldsymbol{Y}_1}(z)=\boldsymbol{Y}_1(z)-\boldsymbol{N}(z)\boldsymbol{J}(z)$, $\overline{\boldsymbol{X}_1}(z)=\boldsymbol{X}_1(z)+\boldsymbol{F}_1(z)\boldsymbol{J}(z)$, then

$$\begin{pmatrix}\boldsymbol{X}_2(z) & -\boldsymbol{Y}_2(z)\\ \boldsymbol{F}_2(z) & \boldsymbol{M}(z)\end{pmatrix}\begin{pmatrix}\boldsymbol{N}(z) & \overline{\boldsymbol{Y}_1}(z)\\ -\boldsymbol{F}_1(z) & \overline{\boldsymbol{X}_1}(z)\end{pmatrix}=\begin{pmatrix}\boldsymbol{I}_{q_1} & 0\\ 0 & \boldsymbol{I}_{p_2}\end{pmatrix}.$$

Note that

$$\begin{pmatrix} \boldsymbol{N}(z) & \boldsymbol{I}_{q_2} - \boldsymbol{N}(z)\boldsymbol{X}_2(z) \\ \boldsymbol{I}_{q_1} & -\boldsymbol{X}_2(z) \end{pmatrix} \begin{pmatrix} \boldsymbol{X}_2(z) & -\boldsymbol{Y}_2(z)\boldsymbol{F}_1(z) \\ \boldsymbol{I}_{q_2} & \boldsymbol{N}(z) \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_{q_2} & 0 \\ 0 & -\boldsymbol{I}_{q_1} \end{pmatrix},$$

therefore

$$\begin{pmatrix} \boldsymbol{X}_2(z) & -\boldsymbol{Y}_2(z) \\ \boldsymbol{F}_2(z) & \boldsymbol{M}(z) \end{pmatrix} \text{and} \begin{pmatrix} \boldsymbol{X}_2(z) & -\boldsymbol{Y}_2(z)\boldsymbol{F}_1(z) \\ \boldsymbol{I}_{q_2} & \boldsymbol{N}(z) \end{pmatrix}$$

are unimodular. Then we have following equation:

$$\begin{pmatrix} \boldsymbol{X}_2(z) & -\boldsymbol{Y}_2(z) \\ \boldsymbol{F}_2(z) & \boldsymbol{M}(z) \end{pmatrix} \begin{pmatrix} \boldsymbol{I}_{q_2} & 0 \\ 0 & \boldsymbol{F}_1(z) \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_{q_1} & 0 \\ 0 & \boldsymbol{F}_2(z) \end{pmatrix} \begin{pmatrix} \boldsymbol{X}_2(z) & -\boldsymbol{Y}_2(z)\boldsymbol{F}_1(z) \\ \boldsymbol{I}_{q_2} & \boldsymbol{N}(z) \end{pmatrix}.$$

Thus, $\begin{pmatrix} \boldsymbol{I}_{q_2} & 0 \\ 0 & \boldsymbol{F}_1(z) \end{pmatrix}$ and $\begin{pmatrix} \boldsymbol{I}_{q_1} & 0 \\ 0 & \boldsymbol{F}_2(z) \end{pmatrix}$ are unimodular equivalent.

Sufficiency: Let following matrices $\begin{pmatrix} \boldsymbol{I}_{q_2} & 0 \\ 0 & \boldsymbol{F}_1(z) \end{pmatrix}$ and $\begin{pmatrix} \boldsymbol{I}_{q_1} & 0 \\ 0 & \boldsymbol{F}_2(z) \end{pmatrix}$ be unimodular equivalent, then there exist two unimodular matrices $\begin{pmatrix} \boldsymbol{X}(z) & \boldsymbol{Y}(z) \\ \boldsymbol{U}(z) & \boldsymbol{M}(z) \end{pmatrix}, \begin{pmatrix} \boldsymbol{L}(z) & \boldsymbol{R}(z) \\ \boldsymbol{W}(z) & \boldsymbol{N}(z) \end{pmatrix}$ such that

$$\begin{pmatrix} \boldsymbol{X}(z) & \boldsymbol{Y}(z) \\ \boldsymbol{U}(z) & \boldsymbol{M}(z) \end{pmatrix} \begin{pmatrix} \boldsymbol{I}_{q_2} & 0 \\ 0 & \boldsymbol{F}_1(z) \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_{q_1} & 0 \\ 0 & \boldsymbol{F}_2(z) \end{pmatrix} \begin{pmatrix} \boldsymbol{L}(z) & \boldsymbol{R}(z) \\ \boldsymbol{W}(z) & \boldsymbol{N}(z) \end{pmatrix}.$$

As can be seen from the above equation, $\boldsymbol{M}(z)\boldsymbol{F}_1(z) = \boldsymbol{F}_2(z)\boldsymbol{N}(z)$, $\boldsymbol{U}(z) = \boldsymbol{F}_2(z)\boldsymbol{W}(z)$.

Since $\begin{pmatrix} \boldsymbol{X}(z) & \boldsymbol{Y}(z) \\ \boldsymbol{F}_2(z)\boldsymbol{W}(z) & \boldsymbol{M}(z) \end{pmatrix}$ is a unimodular matrix, the rank $(\boldsymbol{F}_2(z)\boldsymbol{W}(z), \boldsymbol{M}(z)) = p_2$, this means that $\boldsymbol{F}_2(z)\boldsymbol{W}(z)$, $\boldsymbol{M}(z)$ are ZLC. By Lemma 1, then there exist two matrices $\boldsymbol{A}(z)$, $\boldsymbol{B}(z)$ such that $\boldsymbol{F}_2(z)\boldsymbol{W}(z)\boldsymbol{A}(z) + \boldsymbol{M}(z)\boldsymbol{B}(z) = \boldsymbol{I}_{p_2}$, therefore $\boldsymbol{F}_2(z)$, $\boldsymbol{M}(z)$ are ZLC.

Arguing similarly as in the above proof, we can obtain that $\boldsymbol{F}_1(z)$, $\boldsymbol{N}(z)$ are ZRC. Therefore, $\boldsymbol{F}_1(z)$, $\boldsymbol{F}_2(z)$ are zero coprime equivalent. The proof is completed.

Theorem 2 Let $\boldsymbol{F}_i(z) \in R^{p_i \times q_i}$, $i = 1, 2$ be nD polynomial matrices and $p_1 - q_1 = p_2 - q_2$. If $\begin{pmatrix} \boldsymbol{I}_{q_2+k} & 0 \\ 0 & \boldsymbol{F}_1(z) \end{pmatrix}$ and $\begin{pmatrix} \boldsymbol{I}_{q_1+k} & 0 \\ 0 & \boldsymbol{F}_2(z) \end{pmatrix}$ are unimodular equivalent, where $k \geqslant -\min\{q_1, q_2\}$, then $\boldsymbol{F}_1(z)$ and $\boldsymbol{F}_2(z)$ are zero coprime equivalent.

Proof Suppose that $\begin{pmatrix} \boldsymbol{I}_{q_2+k} & 0 \\ 0 & \boldsymbol{F}_1(z) \end{pmatrix}$ and $\begin{pmatrix} \boldsymbol{I}_{q_1+k} & 0 \\ 0 & \boldsymbol{F}_2(z) \end{pmatrix}$ are unimodular equivalent, then we can obtain that they are zero coprime equivalent. We construct the following equation

$$\boldsymbol{F}_i(z) = \begin{pmatrix} \boldsymbol{I}_{q_j+k} & \\ & \boldsymbol{F}_i(z) \end{pmatrix} \begin{pmatrix} 0 \\ \boldsymbol{I}_{q_i} \end{pmatrix},$$

where $i, j \in \{1, 2\}$ and $i \neq j$.

Obviously, $\boldsymbol{F}_i(z)$ and $\begin{pmatrix} \boldsymbol{I}_{q_j+k} & \\ & \boldsymbol{F}_i(z) \end{pmatrix}$ are zero coprime equivalent. According to the transitivity of zero coprime equivalence, we can further obtain that $\boldsymbol{F}_1(z)$ and $\boldsymbol{F}_2(z)$ are zero coprime equivalent. The proof is completed.

Based on Theorems 1 and 2, we establish the relationship between zero coprime equivalence and unimodular equivalence of nD polynomial matrices. Next, we propose some important results for the zero coprime equivalence of several classes of nD polynomial matrices and their Smith normal forms.

Theorem 3 Let $\boldsymbol{F}(z) \in R^{2\times 2}$ have the following form

$$\boldsymbol{F}(z) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{ij} \in R$, $i, j = 1, 2$ and $d = \det(\boldsymbol{F}(z)) \neq 0$. If all the 1×1 minors of $\boldsymbol{F}(z)$ have a common zero and factor coprime, then

$\boldsymbol{F}(z)$ is not zero coprime equivalent to its Smith normal form $\boldsymbol{S}(z)$, where $\boldsymbol{S}(z)=\begin{pmatrix}1 & 0\\ 0 & d\end{pmatrix}$.

Proof Suppose that there exist $\boldsymbol{M}(z)$, $\boldsymbol{N}(z)\in R^{2\times 2}$ satisfies $\boldsymbol{M}(z)\boldsymbol{F}(z)=\boldsymbol{S}(z)\boldsymbol{N}(z)$, where

$$\boldsymbol{M}(z)=\begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}, \boldsymbol{N}(z)=\begin{pmatrix}n_{11} & n_{12}\\ n_{21} & n_{22}\end{pmatrix} \text{ and } m_{ij}, n_{ij}\in R,\ i,j=1,2.$$

We have

$$\begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}\begin{pmatrix}f_{11} & f_{12}\\ f_{21} & f_{22}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & d\end{pmatrix}\begin{pmatrix}n_{11} & n_{12}\\ n_{21} & n_{22}\end{pmatrix}, \tag{1}$$

then

$$m_{21}f_{11}+m_{22}f_{21}=n_{21}d; \tag{2}$$

$$m_{21}f_{12}+m_{22}f_{22}=n_{22}d. \tag{3}$$

From the equations (2) and (3), we can solve to obtain that $m_{21}=n_{21}f_{22}-n_{22}f_{21}$ and $m_{22}=n_{22}f_{11}-n_{21}f_{12}$. Then we consider all the 2×2 minors of $\begin{pmatrix}\boldsymbol{M}(z) & \boldsymbol{S}(z)\end{pmatrix}$,

$$\{m_{11}m_{22}-m_{12}m_{21},\ -m_{21},\ m_{11}d,\ -m_{22},\ m_{12}d,\ d\}.$$

Suppose that all the 1×1 minors of $F(z)$ have a common zero $\boldsymbol{\alpha}_0=(z_{10}, z_{20}, \cdots, z_{n0})$. It is seen that $m_{21}=m_{22}=0$. So the 2×2 minors of $\begin{pmatrix}\boldsymbol{M}(z) & \boldsymbol{S}(z)\end{pmatrix}$ have a common zero $\boldsymbol{\alpha}_0$. Therefore, $\boldsymbol{F}(z)$ is not zero coprime equivalent to its Smith normal form. The proof is completed.

Remark 1 By Definition 5 and Definition 6, we have that if two nD polynomial matrices are unimodular equivalent, then they must be zero coprime equivalent. Meanwhile, two polynomial matrices that are not zero coprime equivalent imply that they are not unimodular equivalent. Therefore, Theorem 3 can be used to determine whether two matrices are not unimodular equivalent. Furthermore, Proposition 2.9 of Liu *et al*[10] is a special case of Theorem 3.

In addition, we consider nD polynomial matrices $\boldsymbol{F}(z)\in R^{l\times l}$ with the $(l-1)\times(l-1)$ minors generating unit ideal R. In general this kind of matrix $\boldsymbol{F}(x)$ may not be unimodular equivalent to their Smith normal forms, see examples in Frost and Storey[27] or the following matrix,

$$\boldsymbol{A}=\begin{pmatrix}z_1-1 & z_2\\ z_3 & z_1-2\end{pmatrix},$$

by Proposition 2.10 in Ref.[10], $\boldsymbol{A}$ is not unimodular equivalent to its Smith normal form.

Specially, Li *et al*[14] showed that even for the case that

$$\det\boldsymbol{F}(z)=(z_1-f_1(z_2,\cdots,z_n))\cdot(z_1-f_2(z_2,\cdots,z_n)),$$

where f_1, f_2 are different and the $(l-1)\times(l-1)$ minors of $F(z)$ generate R, $\boldsymbol{F}(z)$ may not be unimodular equivalent to its Smith normal form. In what follows, we investigate the conditions under which such matrices are zero coprime equivalent to their Smith normal forms.

Let $\boldsymbol{F}(z)\in R^{2\times 2}$ and

$$\boldsymbol{F}(z)=\begin{pmatrix}z_1-f_1 & p\\ 0 & z_1-f_2\end{pmatrix}, \tag{4}$$

where $f_1, f_2\in R_1$, $p\in R$. According to Definition 2, the Smith normal form of $\boldsymbol{F}(z)$ is

$$\boldsymbol{S}(z)=\begin{pmatrix}1 & 0\\ 0 & (z_1-f_1)(z_1-f_2)\end{pmatrix}.$$

Theorem 4 Let $\boldsymbol{F}(z)$ with form in (4) and $q=z_1-f_1-p$. If q, z_1-f_2 have no common zeros in the field K, then $\boldsymbol{F}(z)$ is zero coprime equivalent to its Smith normal form $\boldsymbol{S}(z)$.

Proof Let

$$\boldsymbol{M}(z)=\begin{pmatrix}m_1 & m_2\\ m_3 & m_4\end{pmatrix}, \quad \boldsymbol{N}(z)=\begin{pmatrix}n_1 & n_2\\ n_3 & n_4\end{pmatrix},$$

where $m_i, n_i\in R$, $i=1,2,3,4$, such that $\boldsymbol{M}(z)\boldsymbol{F}(z)=\boldsymbol{S}(z)\boldsymbol{N}(z)$, i.e.

$$\begin{pmatrix}m_1 & m_2\\ m_3 & m_4\end{pmatrix}\begin{pmatrix}z_1-f_1 & p\\ 0 & z_1-f_2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & (z_1-f_1)(z_1-f_2)\end{pmatrix}\begin{pmatrix}n_1 & n_2\\ n_3 & n_4\end{pmatrix}, \tag{5}$$

obviously, $n_1=m_1(z_1-f_1)$, $n_2=m_1p+m_2(z_1-f_2)$.

Then $m_3(z_1-f_1)=n_3(z_1-f_1)(z_1-f_2)$, and we have $m_3=n_3(z_1-f_2)$. Then $m_3p+m_4(z_1-f_2)=n_4(z_1-f_1)(z_1-f_2)$, from the equation above, we have $n_3p+m_4=n_4(z_1-f_1)$, or equivalently $m_4=n_4(z_1-f_1)-n_3p$.

Now, we consider all the 2×2 minors of matrix $\begin{pmatrix}\boldsymbol{M}(z) & \boldsymbol{S}(z)\end{pmatrix}$:

$$\{m_1m_4-m_2m_3,(z_1-f_1)(z_1-f_2),-m_4,m_1(z_1-f_1)(z_1-f_2),m_2(z_1-f_1)(z_1-f_2),-n_3(z_1-f_2)\}.$$

Let $n_3=n_4=1$, then $m_4=z_1-f_1-p=q$. Since q and z_1-f_2 have no common zeros in the field K, $\boldsymbol{M}(z)$, $\boldsymbol{S}(z)$ are ZLC.

We next prove that $\boldsymbol{F}(z)$, $\boldsymbol{N}(z)$ are ZRC. Since $n_3=n_4=1$, by computing, the 2×2 minors of matrix $\begin{pmatrix}\boldsymbol{F}(z)\\ \boldsymbol{N}(z)\end{pmatrix}$ give

$$\left\{(z_1-f_1)(z_1-f_2),m_2(z_1-f_1)(z_1-f_2),z_1-f_1-p,-m_1(z_1-f_1)(z_1-f_2),-(z_1-f_2),m_1(z_1-f_1)-m_1p-m_2(z_1-f_2)\right\}.$$

Because z_1-f_1-p and $-(z_1-f_2)$ have no common zeros, $\boldsymbol{F}(z)$, $\boldsymbol{N}(z)$ are ZRC. Therefore, $\boldsymbol{F}(z)$ is zero coprime equivalent to its Smith normal form $\boldsymbol{S}(z)$. The proof is completed.

It follows from Theorem 4 that Problem 1 is correct for the case of $l=2$. And then, we extend the conclusions and focus on the case of $\boldsymbol{F}(z)\in R^{l\times l}, l\geqslant 3$.

Theorem 5 Let $\boldsymbol{F}(z)\in R^{3\times 3}$ have the following form

$$\boldsymbol{F}(z)=\begin{pmatrix}1 & a & b\\ 0 & z_1-f_1 & c\\ 0 & 0 & z_1-f_2\end{pmatrix},$$

where $a,b,c\in R$ and $f_1,f_2\in R_1$. Let $q=z_1-f_1-c$. If q,z_1-f_2 have no common zeros in the field K, then $\boldsymbol{F}(z)$ is zero coprime equivalent to its Smith normal form $\boldsymbol{S}(z)$, where

$$\boldsymbol{S}(z)=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & (z_1-f_1)(z_1-f_2)\end{pmatrix}.$$

Proof Let

$$\boldsymbol{M}(z)=\begin{pmatrix}m_1 & m_2 & m_3\\ m_4 & m_5 & m_6\\ 0 & z_1-f_2 & z_1-f_1-c\end{pmatrix},\quad \boldsymbol{N}(z)=\begin{pmatrix}n_1 & n_2 & n_3\\ n_4 & n_5 & n_6\\ 0 & 1 & 1\end{pmatrix},$$

satisfy equation $\boldsymbol{M}(z)\boldsymbol{F}(z)=\boldsymbol{S}(z)\boldsymbol{N}(z)$, where $m_i,n_i\in R, i=1,\cdots,6$.

Now, we consider the 3×3 minors of matrix $\begin{pmatrix}\boldsymbol{M}(z) & \boldsymbol{S}(z)\end{pmatrix}$, where

$$(\boldsymbol{M}(z)\ \boldsymbol{S}(z))=\begin{pmatrix}m_1 & m_2 & m_3 & 1 & 0 & 0\\ m_4 & m_5 & m_6 & 0 & 1 & 0\\ 0 & z_1-f_2 & z_1-f_1-c & 0 & 0 & (z_1-f_1)(z_1-f_2)\end{pmatrix},$$

there exist two 3×3 minors as follows

$$\begin{vmatrix}m_2 & 1 & 0\\ m_5 & 0 & 1\\ z_1-f_2 & 0 & 0\end{vmatrix}=z_1-f_2,\quad \begin{vmatrix}m_3 & 1 & 0\\ m_6 & 0 & 1\\ z_1-f_1-c & 0 & 0\end{vmatrix}=z_1-f_1-c=q.$$

Note that q and z_1-f_2 have no common zeros in the field K, therefore $\boldsymbol{M}(z)$, $\boldsymbol{S}(z)$ are ZLC.

Next, we consider the 3×3 minors of matrix $\begin{pmatrix}\boldsymbol{F}(z)\\ \boldsymbol{N}(z)\end{pmatrix}$, where

$$\begin{pmatrix}\boldsymbol{F}(z)\\ \boldsymbol{N}(z)\end{pmatrix}=\begin{pmatrix}1 & a & b\\ 0 & z_1-f_1 & c\\ 0 & 0 & z_1-f_2\\ n_1 & n_2 & n_3\\ n_4 & n_5 & n_6\\ 0 & 1 & 1\end{pmatrix},$$

similarly, there exist two 3×3 minors as follows

$$\begin{vmatrix} 1 & a & b \\ 0 & 0 & z_1-f_2 \\ 0 & 1 & 1 \end{vmatrix} = -(z_1-f_2), \begin{vmatrix} 1 & a & b \\ 0 & z_1-f_1 & c \\ 0 & 1 & 1 \end{vmatrix} = z_1-f_1-c=q.$$

So the 3×3 minors of matrix $\begin{pmatrix} \boldsymbol{F}(z) \\ \boldsymbol{N}(z) \end{pmatrix}$ have no common zeros in the field K, therefore $\boldsymbol{F}(z)$, $\boldsymbol{N}(z)$ are ZRC. With the help of the above conclusions, $\boldsymbol{F}(z)$ is zero coprime equivalent to its Smith normal form $\boldsymbol{S}(z)$. The proof is completed.

Theorem 6 Let $\boldsymbol{F}(z)\in R^{l\times l}$ have the following form:

$$\boldsymbol{F}(z)=\begin{pmatrix} 1 & f_{12} & \cdots & f_{1,l-1} & f_{1l} \\ 0 & 1 & \cdots & f_{2,l-1} & f_{2l} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & z_1-f_1 & f_{l-1,l} \\ 0 & 0 & \cdots & 0 & z_1-f_2 \end{pmatrix},$$

where $f_{ij}\in R$, $1\leqslant i<j\leqslant l$ and $f_1,f_2\in R_1$. Let $q=z_1-f_1-f_{l-1,l}$. If q, z_1-f_2 have no common zeros in the field K, then $\boldsymbol{F}(z)$ is zero coprime equivalent to its Smith normal form $\boldsymbol{S}(z)$, where

$$\boldsymbol{S}(z)=\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & (z_1-f_1)(z_1-f_2) \end{pmatrix}.$$

Proof Let

$$\boldsymbol{M}(z)=\begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1,l-1} & m_{1l} \\ m_{21} & m_{22} & \cdots & m_{2,l-1} & m_{2l} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{l-1,1} & m_{l-1,2} & \cdots & m_{l-1,l-1} & m_{l-1,l} \\ 0 & 0 & \cdots & z_1-f_2 & q \end{pmatrix}, \boldsymbol{N}(z)=\begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1,l-1} & n_{1l} \\ n_{21} & n_{22} & \cdots & n_{2,l-1} & n_{2l} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n_{l-1,1} & n_{l-1,2} & \cdots & n_{l-1,l-1} & n_{l-1,l} \\ 0 & 0 & \cdots & 1 & 1 \end{pmatrix},$$

where $m_{ij}, n_{ij}\in R$, $1\leqslant i\leqslant l-1$, $1\leqslant j\leqslant l$ and satisfy equation $\boldsymbol{M}(z)\boldsymbol{F}(z)=\boldsymbol{S}(z)\boldsymbol{N}(z)$.

Arguing similarly as in the proof of Theorem 5, it is easy to prove that $\boldsymbol{M}(z)$, $\boldsymbol{S}(z)$ are ZLC and $\boldsymbol{F}(z)$, $\boldsymbol{N}(z)$ are ZRC. Therefore, $\boldsymbol{F}(z)$ is zero coprime equivalent to its Smith normal form $\boldsymbol{S}(z)$. The proof is completed.

Theorem 6 gives a positive answer to Problem 1. Next, we will investigate Problem 2 and develop some new results concerning Theorem 4.

Theorem 7 Let $\boldsymbol{F}(z)$ with form in (4). If p and z_1-f_2 have no common zeros in the field K, then $\boldsymbol{F}(z)$ is zero coprime equivalent to its Smith normal form $\boldsymbol{S}(z)$.

Proof Assume

$$\boldsymbol{M}(z)=\begin{pmatrix} m_1 & m_2 \\ z_1-f_2 & -p \end{pmatrix}, \quad \boldsymbol{N}(z)=\begin{pmatrix} n_1 & n_2 \\ 1 & 0 \end{pmatrix},$$

where $m_i, n_i\in R, i=1,2$ and satisfies equation $\boldsymbol{M}(z)\boldsymbol{F}(z)=\boldsymbol{S}(z)\boldsymbol{N}(z)$,

$$\begin{pmatrix} m_1 & m_2 \\ z_1-f_2 & -p \end{pmatrix}\begin{pmatrix} z_1-f_1 & p \\ 0 & z_1-f_2 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & (z_1-f_1)(z_1-f_2) \end{pmatrix}\begin{pmatrix} n_1 & n_2 \\ 1 & 0 \end{pmatrix}.$$

Arguing similarly as in the proof of Theorem 4, it is also easy to prove that $\boldsymbol{M}(z)$, $\boldsymbol{S}(z)$ are ZLC and $\boldsymbol{F}(z)$, $\boldsymbol{N}(z)$ are ZRC. Therefore, $\boldsymbol{F}(z)$ is zero coprime equivalent to its Smith normal form $\boldsymbol{S}(z)$. The proof is completed.

Theorem 8 Let $\boldsymbol{F}(z)\in R^{3\times 3}$ have the following form $\boldsymbol{F}(z)=\begin{pmatrix} z_1-f_1 & a & b \\ 0 & z_1-f_2 & c \\ 0 & 0 & z_1-f_3 \end{pmatrix}$, where $a,b,c\in R$ and $f_1,f_2,f_3\in R_1$. If the last two columns of $\boldsymbol{F}(z)$ construct a ZRP matrix, then $\boldsymbol{F}(z)$ is zero coprime equivalent to its Smith normal form $\boldsymbol{S}(z)$, where $\boldsymbol{S}(z)=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & (z_1-f_1)(z_1-f_2)(z_1-f_3)\end{pmatrix}$.

Proof Let

$$\boldsymbol{M}(z)=\begin{pmatrix}1 & z_1-f_1 & m_3\\ m_4 & m_5 & m_6\\ (z_1-f_2)(z_1-f_3) & (z_1-f_1)(z_1-f_2)(z_1-f_3)-(z_1-f_3)a & ac-(z_1-f_2)b\end{pmatrix},$$

$$\boldsymbol{N}(z)=\begin{pmatrix}z_1-f_1 & a+(z_1-f_1)(z_1-f_2) & n_3\\ n_4 & n_5 & n_6\\ 1 & z_1-f_2 & c\end{pmatrix},$$

where $m_i, n_i \in R, i=3,\cdots,6$ and satisfy equation $\boldsymbol{M}(z)\boldsymbol{F}(z)=\boldsymbol{S}(z)\boldsymbol{N}(z)$.

Setting $\boldsymbol{P}(z)=\begin{pmatrix}a & b\\ z_1-f_2 & c\\ 0 & z_1-f_3\end{pmatrix}$, from the assumption, $\boldsymbol{P}(z)$ is a ZRP matrix such that all the 2×2 minors $\{ac-b(z_1-f_2),\ (z_1-f_2)(z_1-f_3),\ a(z_1-f_3)\}$ generate unit ideal R.

Now, we consider the 3×3 minors of matrix $\begin{pmatrix}\boldsymbol{M}(z) & \boldsymbol{S}(z)\end{pmatrix}$. By computation, there exist three minors in all the 3×3 minors as follows

$$\{ac-b(z_1-f_2),\ (z_1-f_2)(z_1-f_3),\ -a(z_1-f_3)\}.$$

It is clear that the 3×3 minors of matrix $\begin{pmatrix}\boldsymbol{M}(z) & \boldsymbol{S}(z)\end{pmatrix}$ generate R, therefore $\boldsymbol{M}(z)$, $\boldsymbol{S}(z)$ are ZLC.

Next, we consider the 3×3 minors of matrix $\begin{pmatrix}\boldsymbol{F}(z)\\ \boldsymbol{N}(z)\end{pmatrix}$. Similarly, there exist three minors in all the 3×3 minors as follows

$$\{ac-b(z_1-f_2),\ (z_1-f_2)(z_1-f_3),\ -a(z_1-f_3)\}.$$

So the 3×3 minors of matrix $\begin{pmatrix}\boldsymbol{F}(z)\\ \boldsymbol{N}(z)\end{pmatrix}$ generate R, therefore $\boldsymbol{F}(z)$, $\boldsymbol{N}(z)$ are ZRC. As mentioned earlier, $\boldsymbol{F}(z)$ is zero coprime equivalent to its Smith normal form $\boldsymbol{S}(z)$. The proof is completed.

In what follows, we will focus on the case of $\boldsymbol{F}(z)\in R^{l\times l}$, where $l\geqslant 4$, as presented in the following theorem.

Theorem 9 Let $\boldsymbol{F}(z)\in R^{l\times l}\,(l\geqslant 4)$ have the following form

$$\boldsymbol{F}(z)=\begin{pmatrix}z_1-f_1 & f_{12} & \cdots & f_{1,l-1} & f_{1l}\\ 0 & z_1-f_2 & \cdots & f_{2,l-1} & f_{2l}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & z_1-f_{l-1} & f_{l-1,l}\\ 0 & 0 & \cdots & 0 & z_1-f_l\end{pmatrix},$$

where $f_1,\cdots,f_l\in R_1$, $f_{ij}\in R$, $1\leqslant i<j\leqslant l$. If the last $l-1$ columns of $\boldsymbol{F}(z)$ construct a ZRP matrix, then $\boldsymbol{F}(z)$ is zero coprime equivalent to its Smith normal form $\boldsymbol{S}(z)$, where

$$\boldsymbol{S}(z)=\begin{pmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & \cdots & 0 & (z_1-f_1)(z_1-f_2)\cdots(z_1-f_l)\end{pmatrix}.$$

Proof We first prove the case of $l=4$. Let

$$\boldsymbol{P}(z)=\begin{pmatrix}f_{12} & f_{13} & f_{14}\\ z_1-f_2 & f_{23} & f_{24}\\ 0 & z_1-f_3 & f_{34}\\ 0 & 0 & z_1-f_4\end{pmatrix},$$

by computing, $\det \boldsymbol{F}(z)=d=(z_1-f_1)(z_1-f_2)(z_1-f_3)(z_1-f_4)$, the 3×3 minors of matrix $\boldsymbol{P}(z)$ as follows:

$$c_1=(z_1-f_2)(z_1-f_3)(z_1-f_4), c_2=f_{12}(z_1-f_3)(z_1-f_4),$$

$$c_3=(z_1-f_4)[f_{12}f_{23}-f_{13}(z_1-f_2)], c_4=f_{12}f_{23}f_{34}+f_{14}(z_1-f_2)(z_1-f_3)-f_{12}f_{24}(z_1-f_3)-f_{13}f_{34}(z_1-f_2).$$

From the assumption that matrix $\boldsymbol{P}(z)$ is ZRP, we have that c_1, c_2, c_3, c_4 generate unit ideal R.

Let

$$\boldsymbol{M}(z)=\begin{pmatrix} 1 & z_1-f_1 & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ c_1 & d-c_2 & c_3 & c_4 \end{pmatrix},$$

$$\boldsymbol{N}(z)=\begin{pmatrix} z_1-f_1 & f_{12}+(z_1-f_1)(z_1-f_2) & n_{13} & n_{14} \\ n_{21} & n_{22} & n_{23} & n_{24} \\ n_{31} & n_{32} & n_{33} & n_{34} \\ 1 & z_1-f_2 & f_{23} & f_{24} \end{pmatrix},$$

satisfy the equation $\boldsymbol{M}(z)\boldsymbol{F}(z)=\boldsymbol{S}(z)\boldsymbol{N}(z)$, where $m_{ij}, n_{ij}\in R$.

Next we consider the 4×4 minors of matrix $\begin{pmatrix}\boldsymbol{M}(z) & \boldsymbol{S}(z)\end{pmatrix}$. Since the polynomials $-c_1, -c_2, -c_3, -c_4$ belong to the 4×4 minors of matrix $\begin{pmatrix}\boldsymbol{M}(z) & \boldsymbol{S}(z)\end{pmatrix}$, $\boldsymbol{M}(z)$, $\boldsymbol{S}(z)$ are ZLC. Similarly, we obtain that $\boldsymbol{F}(z)$, $\boldsymbol{N}(z)$ are ZRC. Therefore, $\boldsymbol{F}(z)$ is zero coprime equivalent to its Smith normal form $\boldsymbol{S}(z)$.

Using a process similar to the above proof, we can straightforwardly get the result for the case of $l>4$. Therefore, $\boldsymbol{F}(z)$ is zero coprime equivalent to its Smith normal form $\boldsymbol{S}(z)$ for $l\geqslant 4$. The proof is completed.

Remark 2 Combining Theorem 7, Theorem 8 and Theorem 9, we give a positive answer to Problem 2.

3 Example

In this section, we give an example to illustrate the main results.

Example 1 Consider a 2D polynomial matrix of $R^{5\times 5}$

$$\boldsymbol{F}(z,w)=\begin{pmatrix} z-w^2 & -z+2w-1 & z^2-zw-z+w-1 & 0 & 1 \\ 0 & z-2w & z^2 & -z-1 & 0 \\ 0 & 0 & z-1 & -1 & 0 \\ 0 & 0 & 0 & z-w & 1 \\ 0 & 0 & 0 & 0 & z-w^2+w \end{pmatrix}.$$

By computing, let $d=\det \boldsymbol{F}(z,w)=(z-w^2)(z-2w)(z-1)(z-w)(z-w^2+w)$ and the last 4 columns of $\boldsymbol{F}(z,w)$ construct a ZRP matrix. Obviously, the Smith normal forms of $\boldsymbol{F}(z,w)$ is following matrix

$$\boldsymbol{S}(z,w)=\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & d \end{pmatrix}.$$

Let

$$\boldsymbol{P}(z,w)=\begin{pmatrix} -z+2w-1 & z^2-zw-z+w-1 & 0 & 1 \\ z-2w & z^2 & -z-1 & 0 \\ 0 & z-1 & -1 & 0 \\ 0 & 0 & z-w & 1 \\ 0 & 0 & 0 & z-w^2+w \end{pmatrix},$$

calculating all the 4×4 minors of matrix $\boldsymbol{P}(z,w)$ as follows:

$c_1=(z-2w)(z-1)(z-w)(z-w^2+w),\ c_2=(-z+2w-1)(z-1)(z-w)(z-w^2+w),$

$c_3=z^2(-z+2w-1)(z-w)(z-w^2+w)-(z-2w)(z^2-zw-z+w-1)(z-w)(z-w^2+w),$

$c_4=-(z-w^2+w)[z^2(-z+2w-1)-(z-2w)(z^2-zw-z+w-1)+(z-1)(-z-1)(-z+2w-1)],\ c_5=1.$

Setting

$$\boldsymbol{M}(z,w)=\begin{pmatrix} 1 & z-w^2 & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} \\ c_1 & d-c_2 & c_3 & -c_4 & c_5 \end{pmatrix},$$

$$\boldsymbol{N}(z,w)=\begin{pmatrix} z-w^2 & -z+2w-1+(z-w^2)(z-2w) & n_{13} & n_{14} & n_{15} \\ n_{21} & n_{22} & n_{23} & n_{24} & n_{25} \\ n_{31} & n_{32} & n_{33} & n_{34} & n_{35} \\ n_{41} & n_{42} & n_{43} & n_{44} & n_{45} \\ 1 & z-2w & z^2 & -z-1 & 0 \end{pmatrix},$$

where $m_{ij}, n_{ij} \in R$, such that satisfy equation $\boldsymbol{M}(z,w)\boldsymbol{F}(z,w)=\boldsymbol{S}(z,w)\boldsymbol{N}(z,w)$, we consider that the 5×5 minors of matrix $\begin{pmatrix}\boldsymbol{M}(z,w) & \boldsymbol{S}(z,w)\end{pmatrix}$ have a fifth order minor which is equal to 1, therefore, $\boldsymbol{M}(z,w)$, $\boldsymbol{S}(z,w)$ are ZLC. Similarly, by calculating the 5×5 minors of matrix $\begin{pmatrix}\boldsymbol{F}(z,w) \\ \boldsymbol{N}(z,w)\end{pmatrix}$, we find that there is also a fifth order minor which is equal to 1, then $\boldsymbol{F}(z,w)$, $\boldsymbol{N}(z,w)$ are ZRC. Thus, $\boldsymbol{F}(z,w)$ is zero coprime equivalent to its Smith normal form $\boldsymbol{S}(z,w)$.

4 Conclusion

In this paper, we first propose some criteria for the conversion of zero coprime equivalence into unimodular equivalence through trivial expansion. Then we investigate the zero coprime equivalence problem of several kinds of nD polynomial matrices over an algebraic closed field. In general, it is observed that certain multivariate polynomial triangular matrices are not zero coprime equivalent to their Smith normal forms (Theorem 3). Afterwards, we are devoted to studying the zero coprime equivalence and reduction of several classes of multivariate polynomial matrices that are not unimodular equivalent to their Smith normal forms. We present two problems in this aspect and given positive answers to them. An illustrative example has also been comprehensively analyzed towards the end.

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关于多元多项式矩阵的零互素等价

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摘要：零互素系统等价是多维系统等价理论中的重要研究内容之一，与多元多项式矩阵的零互素等价密切相关。本文首先讨论了多项式矩阵的零互素等价和幺模等价之间的关系。然后，我们研究了几类多项式矩阵的零互素等价问题，得到了将这些矩阵简化为其Smith型的一些新发现和准则。最后，通过一个例子来说明主要结果。

关键词：多维系统；多元多项式矩阵；零互素等价；幺模等价；Smith标准型

□