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Time Decay of Linearized Isentropic 2D Bipolar Navier-Stokes-Poisson System

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Abstract: Cauchy problem for the linearized bipolar isentropic Navier-Stokes-Poisson system in $\mathbb{R}^2$ is studied. Through the reformulation of unknown functions, we change the formal system into a linearized Navier-Stokes system and a unipolar Navier-Stokes-Poisson system. Based on a delicate analysis of the corresponding Green function, L^2 decay estimate of the solution is obtained.

Key words: bipolar Navier-Stokes-Poisson system; Green function; L^2 decay

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0 Introduction

Cauchy problem of the two-dimensional bipolar Navier-Stokes-Poisson (BNSP) system has the following formulation:

$$\begin{cases} \rho_t + \operatorname{div} m = 0, \\ m_t + \operatorname{div}\left(\dfrac{m \otimes m}{\rho}\right) + \nabla P_1(\rho) = \mu_1 \Delta\left(\dfrac{m}{\rho}\right) + \mu_2 \nabla \operatorname{div}\left(\dfrac{m}{\rho}\right) + Z\rho e \nabla\phi, \\ n_t + \operatorname{div}\omega = 0, \\ \omega_t + \operatorname{div}\left(\dfrac{\omega \otimes \omega}{n}\right) + \nabla P_2(n) = \bar{\mu}_1 \Delta\left(\dfrac{\omega}{n}\right) + \bar{\mu}_2 \nabla \operatorname{div}\left(\dfrac{\omega}{n}\right) - ne\nabla\phi, \\ \Delta\phi = 4\pi e(Z\rho - n), \quad \lim\limits_{|x| \to \infty} \phi = 0, \\ (\rho, m, n, \omega)(x, 0) = (\rho_0, m_0, n_0, \omega_0)(x), \quad (x, t) \in (\mathbb{R}^2 \times \mathbb{R}^+). \end{cases} \tag{1}$$

Here the unknown functions ρ, n represent the density of ions and electrons, $m(x,t)$, $\omega(x,t)$ are the momentum of ions and electrons, ϕ is the electrostatic potential. ∇ and div are the usual gradient and the divergence operator. μ_1, μ_2, $\bar{\mu}_1$, $\bar{\mu}_2$ are constant positive viscosity coefficients. Pressure functions $P_1(\rho)$, $P_2(n)$ have positive derivatives. The electrons have charge $-e$ and the ions have charge Ze where Z, e are positive constants. For simplicity, we set $e=1$.

The BNSP system is used to describe the dynamics of two separate compressible fluids of ions and electrons with

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their self-consistent electromagnetic field. It is a hyperbolic parabolic coupling system. Because of its physical importance and mathematical challenges, there were extensive studies on the asymptotic and global existence of the BNSP system. For example, Refs. [1-5] dealt with different forms of BNSP, and got the global existence of classical solution and its decay. But most results are about space dimension $n\geqslant 3$. There are few results about space $n=2$. This paper studies L^2 decay estimate of a linearized two-dimensional BNSP system.

Throughout this paper, $L^p(U)$ denotes the Lebesgue integrable space function, $H^p(U)$ means the Sobolev space function. C and C_i denote some general positive constants.

1 Reformulation and Linearization

Suppose the initial value $(\rho_0, m_0, n_0, \omega_0)(x)$ of (1) tends to equilibrium state $\left(\frac{\bar\rho}{Z}, 0, \bar\rho, 0\right)$ as $|x|\to\infty$. Set $\tilde\rho = Z\rho - \bar\rho$, $\tilde m = m - 0$, $\tilde n = n - \bar\rho$, $\tilde\omega = \omega$. Then (1) can be rewritten as

$$\begin{cases} \dfrac{\tilde\rho_t}{Z} + \mathrm{div}\tilde m = 0, \\ \tilde m_t + \mathrm{div}\left(\dfrac{\tilde m \otimes \tilde m}{\frac{\tilde\rho+\bar\rho}{Z}}\right) + \nabla P_1\left(\dfrac{\tilde\rho+\bar\rho}{Z}\right) = \mu_1 \Delta\left(\dfrac{\tilde m}{\frac{\tilde\rho+\bar\rho}{Z}}\right) + \mu_2 \nabla\mathrm{div}\left(\dfrac{\tilde m}{\frac{\tilde\rho+\bar\rho}{Z}}\right) + Z\cdot\dfrac{\tilde\rho+\bar\rho}{Z}\nabla\phi, \\ \tilde n_t + \mathrm{div}\tilde\omega = 0, \\ \tilde\omega_t + \mathrm{div}\left(\dfrac{\tilde\omega\otimes\tilde\omega}{\tilde n+\bar\rho}\right) + \nabla P_2(\tilde n+\bar\rho) = \bar\mu_1\Delta\left(\dfrac{\tilde\omega}{\tilde n+\bar\rho}\right) + \bar\mu_2\nabla\mathrm{div}\left(\dfrac{\tilde\omega}{\tilde n+\bar\rho}\right) - (\tilde n+\bar\rho)\nabla\phi, \\ \Delta\phi = 4\pi(\tilde\rho - \tilde n). \end{cases} \tag{2}$$

System (2) can be reformulated as a linear part plus a nonlinear part.

$$\begin{cases} \tilde\rho_t + Z\mathrm{div}\tilde m = 0, \\ \tilde m_t + P_1'\left(\dfrac{\bar\rho}{Z}\right)\nabla\left(\dfrac{\tilde\rho}{Z}\right) - \mu_1 Z\cdot\dfrac{1}{\bar\rho}\Delta\tilde m - \mu_2 Z\cdot\dfrac{1}{\bar\rho}\nabla\mathrm{div}\tilde m - \bar\rho\nabla\phi = F_1, \\ \tilde n_t + \mathrm{div}\tilde\omega = 0, \\ \tilde\omega_t + P_2'(\bar\rho)\nabla\tilde n - \dfrac{\bar\mu_1}{\bar\rho}\Delta\tilde\omega - \dfrac{\bar\mu_2}{\bar\rho}\nabla\mathrm{div}\tilde\omega + \bar\rho\nabla\phi = F_2. \end{cases} \tag{3}$$

The left side of (3) is the linearized part of (2) near $\left(\frac{\bar\rho}{Z}, 0, \bar\rho, 0\right)$, and the right side of (3) is the nonlinearized part. For simplicity, we denote the perturbation $\tilde\rho$, $\tilde m$, $\tilde n$, $\tilde\omega$ as $\frac{\rho}{Z}, m, n, \omega$, so the linearized system of (1) near the state $\left(\frac{\bar\rho}{Z}, 0, \bar\rho, 0\right)$ is

$$\begin{cases} \rho_t + \mathrm{div}m = 0, \\ m_t + c_1^2\nabla\rho - \dfrac{\mu_1 Z}{\bar\rho}\Delta m - \dfrac{\mu_2 Z}{\bar\rho}\nabla\mathrm{div}m - \bar\rho\nabla\phi = 0, \\ n_t + \mathrm{div}\omega = 0, \\ \omega_t + c_2^2\nabla n - \dfrac{\bar\mu_1}{\bar\rho}\Delta\omega - \dfrac{\bar\mu_2}{\bar\rho}\nabla\mathrm{div}\omega + \bar\rho\nabla\phi = 0, \\ \Delta\phi = 4\pi(\rho - n), \\ (\rho, m, n, \omega)(x, 0) = (\rho_0, m_0, n_0, \omega_0)(x), \quad (x, t)\in(\mathbb{R}^2\times\mathbb{R}^+), \end{cases} \tag{4}$$

where $c_1^2 = P_1'\left(\frac{\bar\rho}{Z}\right)$, $c_2^2 = P_2'(\bar\rho)$.

Our final result in this paper is the following theorem.

Theorem 1 If the initial data $\rho_0-\dfrac{\bar{\rho}}{Z}$, $n_0-\bar{\rho}$, m_0, $w_0\in L^1\cap H^2$, we have the following estimate:

$$\left\|\rho+n-\frac{\bar{\rho}}{Z}-\bar{\rho}\right\|_{L^2}+\|m+\omega\|_{L^2}\leqslant C(1+t)^{-\frac{1}{2}}\left(\left\|\rho_0+n_0-\frac{\bar{\rho}}{Z}-\bar{\rho}\right\|_{L^1}+\left\|\rho_0+n_0-\frac{\bar{\rho}}{Z}-\bar{\rho}\right\|_{L^2}\right)+C(1+t)^{-\frac{1}{2}}\left(\|m_{20}\|_{L^1}+\|m_{20}\|_{L^2}\right),$$

$$\min\left(\left\|\rho+n-\frac{\bar{\rho}}{Z}-\bar{\rho}\right\|_{L^2},\ \|m+\omega\|_{L^2}\right)\geqslant C(1+t)^{-\frac{1}{2}}.$$

If $\left|\hat{\rho}_0-\hat{n}_0\right|_{|\xi|\leqslant\varepsilon}\leqslant C_1|\xi|^{\varepsilon_1}$, for any positive ε_1, we have $\|m-\omega\|_{L^2}\leqslant Ct^{-\varepsilon_1}+C(1+t)^{-\frac{1}{2}}\left(\|m_0-\omega_0\|_{L^1}+\|m_0-\omega_0\|_{H^2}\right)$,

$$\left\|\rho-\frac{\bar{\rho}}{Z}-n+\bar{\rho}\right\|_{L^2}\leqslant C(1+t)^{-\frac{1}{2}}\left(\left\|\rho_0-\frac{\bar{\rho}}{Z}-n_0+\bar{\rho}\right\|_{L^1}+\left\|\rho_0-\frac{\bar{\rho}}{Z}-n_0+\bar{\rho}\right\|_{H^2}\right).$$

Further on, if $\left|\hat{\rho}_0-\hat{n}_0\right|_{|\xi|\leqslant\varepsilon}\geqslant C_2|\xi|^{\varepsilon_1}$, for any positive ε_1, we have $\|m-\omega\|_{L^2}\geqslant C_3t^{-\varepsilon_1}$, $\left\|\rho-\dfrac{\bar{\rho}}{Z}-n+\bar{\rho}\right\|_{L^2}\geqslant C_3(1+t)^{-\frac{1}{2}}$.

Remark 1 From Theorem 1, the perturbation of the sum of density and momentum decay at the rate $(1+t)^{-\frac{1}{2}}$, the perturbation of the difference of density decay at the rate $(1+t)^{-\frac{1}{2}}$, but the difference of momentum hardly decay at t because of the influence of the electronic field. Due to the low decay rate, it is difficult to go on the global existence of the system.

We want to separate system (4) into several small sets of equations. Considering the fifth equation of (4), we denote $\rho_1=\rho+n$, $\rho_2=\rho-n$, $m_1=m+\omega$, $m_2=m-\omega$, (4) is equal to the following system:

$$\begin{cases}\rho_{1t}+\mathrm{div}m_1=0,\\ m_{1t}+\dfrac{c_1^2+c_2^2}{2}\nabla\rho_1+\dfrac{c_1^2-c_2^2}{2}\nabla\rho_2-\dfrac{1}{2}\left(\dfrac{\mu_1Z+\bar{\mu}_1}{\bar{\rho}}\right)\Delta m_1-\dfrac{1}{2}\left(\dfrac{\mu_1Z-\bar{\mu}_1}{\bar{\rho}}\right)\Delta m_2\\ -\dfrac{1}{2}\left(\dfrac{\mu_2Z+\bar{\mu}_2}{\bar{\rho}}\right)\nabla\mathrm{div}m_1-\dfrac{1}{2}\left(\dfrac{\mu_2Z-\bar{\mu}_2}{\bar{\rho}}\right)\nabla\mathrm{div}m_2=0,\\ \rho_{2t}+\mathrm{div}m_2=0,\\ m_{2t}+\dfrac{c_1^2-c_2^2}{2}\nabla\rho_1+\dfrac{c_1^2+c_2^2}{2}\nabla\rho_2-\dfrac{1}{2}\left(\dfrac{\mu_1Z-\bar{\mu}_1}{\bar{\rho}}\right)\Delta m_1-\dfrac{1}{2}\left(\dfrac{\mu_1Z+\bar{\mu}_1}{\bar{\rho}}\right)\Delta m_2\\ -\dfrac{1}{2}\left(\dfrac{\mu_2Z-\bar{\mu}_2}{\bar{\rho}}\right)\nabla\mathrm{div}m_1-\dfrac{1}{2}\left(\dfrac{\mu_2Z+\bar{\mu}_2}{\bar{\rho}}\right)\nabla\mathrm{div}m_2-2\bar{\rho}\Delta\phi=0,\\ \Delta\phi=4\pi\rho_2.\end{cases}\tag{5}$$

For simplicity, we suppose $c_1^2=c_2^2$, $\mu_1Z=\bar{\mu}_1$, $\mu_2Z=\bar{\mu}_2$, denote $c^2=\dfrac{c_1^2+c_2^2}{2}$, $\gamma_1=\dfrac{1}{2}\left(\dfrac{\mu_1Z+\bar{\mu}_1}{\bar{\rho}}\right)$, $\gamma_2=\dfrac{1}{2}\left(\dfrac{\mu_2Z+\bar{\mu}_2}{\bar{\rho}}\right)$. System (5) can be separated into the following two systems

$$\begin{cases}\rho_{1t}+\mathrm{div}m_1=0,\\ m_{1t}+c^2\nabla\rho_1-\gamma_1\Delta m_1-\gamma_2\nabla\mathrm{div}m_1=0.\end{cases}\tag{6}$$

$$\begin{cases}\rho_{2t}+\mathrm{div}m_2=0,\\ m_{2t}+c^2\nabla\rho_2-\gamma_1\Delta m_2-\gamma_2\nabla\mathrm{div}m_2-2\bar{\rho}\nabla\phi=0,\\ \Delta\phi=4\pi\rho_2.\end{cases}\tag{7}$$

We find that (6) is a linearized isentropic Navier-Stokes (NS) system while (7) is a linearized unipolar Navier-Stokes-Poisson (NSP) system. We study them respectively in the next step.

2 L^2 Decay of Linearized NS System

If the initial data of (6) is $(\rho_1,m_1)(x,0)=(\rho_{10},m_{10})(x)$, according to (1.3) in Ref. [6], we have

$$\begin{pmatrix}\hat{\rho}_1\\ \hat{m}_1\end{pmatrix}=\begin{pmatrix}\dfrac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-} & \dfrac{e^{\lambda_-t}-e^{\lambda_+t}}{\lambda_+-\lambda_-}i\xi^{\tau}\\ -ic^2\xi\cdot\dfrac{e^{\lambda_-t}-e^{\lambda_+t}}{\lambda_+-\lambda_-} & \dfrac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-}\dfrac{\xi\xi^{\tau}}{|\xi|^2}+e^{-\gamma_1|\xi|^2t}\left(\boldsymbol{I}-\dfrac{\xi\xi^{\tau}}{|\xi|^2}\right)\end{pmatrix}\begin{pmatrix}\hat{\rho}_{10}\\ \hat{m}_{10}\end{pmatrix}\tag{8}$$

where $\boldsymbol{I}$ is a 2×2 unit matrix,

$$\lambda_+=\frac{-(\gamma_1+\gamma_2)|\xi|^2+\sqrt{(\gamma_1+\gamma_2)^2|\xi|^4-4c^2|\xi|^2}}{2},$$
$$\lambda_-=\frac{-(\gamma_1+\gamma_2)|\xi|^2-\sqrt{(\gamma_1+\gamma_2)^2|\xi|^4-4c^2|\xi|^2}}{2}.\tag{9}$$

From (9), when $|\xi|$ is small enough, there is no problem with the decay estimate for the Green function; when $|\xi|$ is bounded and away from zero, $\lambda_+-\lambda_-$ may tend to zero, so we need to consider the integrability of the Green function; when $|\xi|$ is large enough, for example, $\frac{\xi}{\lambda_+-\lambda_-}=O(\frac{1}{|\xi|})$ has no L_2 bounds. Next, we will divide frequency into three different parts and will use different methods to consider the decay rate respectively.

When $|\xi|^2\leqslant\varepsilon\leqslant\frac{2c^2}{(\gamma_1+\gamma_2)^2}$, we have

$$\frac{\lambda_+e^{\lambda_-t}}{\lambda_+-\lambda_-}=\left(-\frac{(\gamma_1+\gamma_2)|\xi|^2}{2}+ic|\xi|+o\left(|\xi|^2\right)\right)\left(\frac{1}{2ic|\xi|}+o\left(|\xi|\right)\right)e^{\left(-\frac{(\gamma_1+\gamma_2)|\xi|^2}{2}-ic|\xi|+o\left(|\xi|^2\right)\right)t}.$$

We find the construction of λ_1 is like that of λ_2 in Ref. [7]. Initiated by the estimation method in Ref. [7], we get

$$\left\|\frac{\lambda_+e^{\lambda_-t}}{\lambda_+-\lambda_-}\right\|_{L^2\left(|\xi|^2\leqslant\varepsilon\right)}\leqslant C\left\|e^{-\frac{(\gamma_1+\gamma_2)|\xi|^2t}{4}}\right\|_{L^2\left(|\xi|^2\leqslant\varepsilon\right)}\leqslant C\left(\int_0^1e^{-\frac{(\gamma_1+\gamma_2)^2\rho^2t}{2}}\rho\mathrm{d}\rho\right)^{\frac{1}{2}}\leqslant C(1+t)^{-\frac{1}{2}}.$$

Similarly

$$\left\|\frac{e^{\lambda_-t}-e^{\lambda_+t}}{\lambda_+-\lambda_-}i\xi^{\tau}\right\|_{L^2\left(|\xi|^2\leqslant\varepsilon\right)}\leqslant C(1+t)^{-\frac{1}{2}}.$$

Thus

$$\|\hat{\rho}_1\|_{L^2\left(|\xi|^2\leqslant\varepsilon\right)}=\left\|\frac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-}\right\|_{L^2}\|\rho_{10}\|_{L^1}+\left\|\frac{e^{\lambda_-t}-e^{\lambda_+t}}{\lambda_+-\lambda_-}i\xi^{\tau}\right\|_{L^2}\|m_{10}\|_{L^1}\leqslant C(1+t)^{-\frac{1}{2}}\max\left(\|\rho_{10}\|_{L^1},\|m_{10}\|_{L^1}\right)\tag{10}$$

Using the same method, we can get

$$\|\hat{m}_1\|_{L^2\left(|\xi|^2\leqslant\varepsilon\right)}\leqslant C(1+t)^{-\frac{1}{2}}\max\left(\|\rho_{10}\|_{L^1},\|m_{10}\|_{L^1}\right).\tag{11}$$

When $\varepsilon\leqslant|\xi|^2\leqslant\frac{8c^2}{(\gamma_1+\gamma_2)^2}\leqslant R$, there exist positive constant b such that $\lambda_\pm\leqslant-b<0$.

Because $\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}=e^{\lambda_-t}\frac{e^{(\lambda_+-\lambda_-)t}-1}{\lambda_+-\lambda_-}$, $\frac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-}=\lambda_+e^{\lambda_-t}\frac{e^{(\lambda_+-\lambda_-)t}-1}{\lambda_+-\lambda_-}+e^{\lambda_-t}$ are smooth functions, we can easily get

$$\|\hat{\rho}_1\|_{L^2\left(\varepsilon\leqslant|\xi|^2\leqslant R\right)},\ \|\hat{m}_1\|_{L^2\left(\varepsilon\leqslant|\xi|^2\leqslant R\right)}\leqslant Ce^{-bt}\left(\|\rho_{10}\|_{L^1}+\|m_{10}\|_{L^1}\right).\tag{12}$$

When $|\xi|^2\geqslant R$, notice $\lambda_\pm\leqslant-b$ for some positive b, then

$$\frac{1}{\lambda_+-\lambda_-}=\frac{1}{|\xi|^2(\gamma_1+\gamma_2)}\left(1+\frac{2c^2}{(\gamma_1+\gamma_2)^2}\frac{1}{|\xi|^2}+o\left(\frac{1}{|\xi|^2}\right)\right)\leqslant C\frac{1}{|\xi|^2},$$

we have

$$\left\|\hat{G}\begin{pmatrix}\hat{\rho}_{10}\\ \hat{m}_{10}\end{pmatrix}\right\|_{L^2\left(|\xi|^2\geqslant R\right)}\leqslant Ce^{-bt}\left(\|\hat{\rho}_{10}\|_{L^2\left(|\xi|^2\geqslant R\right)}+\|\hat{m}_{10}\|_{L^2\left(|\xi|^2\geqslant R\right)}\right).\tag{13}$$

Next, we consider $\left\| \frac{e^{\lambda_- t}-e^{\lambda_+ t}}{\lambda_+-\lambda_-} i\xi^\tau \right\|_{L^2(|\xi|^2\leqslant\varepsilon)}$.

Since $\sqrt{4c^2|\xi|^2-(\gamma_1+\gamma_2)^2|\xi|^4}=2c|\xi|\cdot\sqrt{1-\frac{(\gamma_1+\gamma_2)}{4c^2}|\xi|^2}=2c|\xi|+O\left(|\xi|^3\right)$, we have

$$\left|\sin 2c|\xi|t\right|\leqslant\left|\sin 2c|\xi|t-\sin\sqrt{4c^2|\xi|^2-(\gamma_1+\gamma_2)^2|\xi|^4}\,t\right|+\left|\sin\sqrt{4c^2|\xi|^2-(\gamma_1+\gamma_2)^2|\xi|^4}\,t\right|$$
$$\leqslant\left|\left(O\left(|\xi|^3\right)t\right)\right|+\left|\sin\sqrt{4c^2|\xi|^2-(\gamma_1+\gamma_2)^2|\xi|^4}\,t\right|,$$

then

$$\sin^2\sqrt{4c^2|\xi|^2-(\gamma_1+\gamma_2)^2|\xi|^4}\,t\geqslant\frac{1}{2}\sin^2 2c|\xi|t+\left(O\left(|\xi|^3\right)t\right)^2,\tag{14}$$

From (9) and (14), we have

$$\begin{aligned}\left\|\frac{e^{\lambda_- t}-e^{\lambda_+ t}}{\lambda_+-\lambda_-}i\xi^\tau\right\|_{L^2(|\xi|^2\leqslant\varepsilon)}&\geqslant C\left\|e^{-\frac{(\gamma_1+\gamma_2)|\xi|^2t}{2}}\sin\sqrt{4c^2|\xi|^2-(\gamma_1+\gamma_2)^2|\xi|^4}\,t\right\|_{L^2(|\xi|^2\leqslant\varepsilon)}\\&\geqslant C\left(\int_{|\xi|^2\leqslant\varepsilon}e^{-\frac{(\gamma_1+\gamma_2)|\xi|^2t}{2}}\cdot\frac{1}{2}\sin^2 2c|\xi|t\mathrm{d}\xi\right)^{\frac{1}{2}}-C\left(\int_{|\xi|^2\leqslant\varepsilon}e^{-\frac{(\gamma_1+\gamma_2)|\xi|^2t}{2}}\left(O\left(|\xi|^3\right)t\right)^2\mathrm{d}\xi\right)^{\frac{1}{2}}=\boldsymbol{I}_1+\boldsymbol{I}_2.\end{aligned}\tag{15}$$

If we fix $t\geqslant\frac{5\pi}{8c\sqrt{\varepsilon}}$, we have

$$\boldsymbol{I}_1\geqslant C\left(\int_{|\xi|\leqslant\sqrt{\varepsilon t}}\int e^{-(\gamma_1+\gamma_2)|\xi|^2}\sin^2(2c|\xi|\sqrt{t})\cdot|\xi|t^{-1}\mathrm{d}|\xi|\right)^{\frac{1}{2}}\geqslant C\cdot\sum_{k=0}^{\left[\frac{2c\sqrt{\varepsilon}t}{\pi}-\frac{1}{4}\right]}\frac{1}{2}\cdot t^{-\frac{1}{2}}\left(\int_{\frac{k\pi+\frac{\pi}{4}}{2c\sqrt{t}}}^{\frac{k\pi+\frac{3\pi}{4}}{2c\sqrt{t}}}e^{-(\gamma_1+\gamma_2)r^2}\cdot r\mathrm{d}r\right)^{\frac{1}{2}}=Ct^{-\frac{1}{2}},\tag{16}$$

$$\boldsymbol{I}_2\leqslant C\left(\int e^{-(\gamma_1+\gamma_2)|\xi|^2}\cdot O\left(|\xi|^3\right)\cdot t^{-\frac{3}{2}}|\xi|\cdot t^{-1}\mathrm{d}|\xi|\right)^{\frac{1}{2}}\leqslant Ct^{-\frac{5}{4}}.\tag{17}$$

From (15), (16), and (17) , when t is large enough, we have

$$\left\|\frac{e^{\lambda_- t}-e^{\lambda_+ t}}{\lambda_+-\lambda_-}i\xi^\tau\right\|_{L^2(|\xi|^2\leqslant\varepsilon)}\geqslant Ct^{-\frac{1}{2}}.\tag{18}$$

From (8) and (18), we have

$$\|\hat{\rho}_1\|_{L^2}\geqslant\|\hat{\rho}_1\|_{L^2(|\xi|\leqslant\varepsilon)}\geqslant C_1t^{-\frac{1}{2}}\|m_{10}\|_{L^1},\tag{19}$$

$$\|\hat{m}_1\|_{L^2}\geqslant\|\hat{m}_1\|_{L^2(|\xi|\leqslant\varepsilon)}\geqslant C_1t^{-\frac{1}{2}}\|\rho_{10}\|_{L^1},\tag{20}$$

Together with (8), (10), (11), (12), (13), (19), and (20), we get our results.

Theorem 2 Suppose $E=\max\left(\|\rho_{10}\|_{L^1},\|m_{10}\|_{L^1},\|\rho_{10}\|_{L^2},\|m_{10}\|_{L^2}\right)$, there exists a positive constant C such that $\max\left(\|\rho_1\|_{L^2},\|m_1\|_{L^2}\right)\leqslant C(1+t)^{-\frac{1}{2}}E$. Further on, when t is large enough, we have $\min(\|\rho_1\|_{L^2},\|m_1\|_{L^2})\geqslant CEt^{-\frac{1}{2}}$.

Remark 2 From Theorem 2, we know our decay estimate about t is optimal.

3 L^2 Decay of Linearized NSP System

Suppose the initial data of (7) is (ρ_{20},m_{20}), using the method of Ref. [8], the solution of (7) can be expressed as

$$\hat{\rho}_2(\xi,t)=\frac{\eta_+e^{\eta_- t}-\eta_-e^{\eta_+ t}}{\eta_+-\eta_-}\hat{\rho}_{20}+\frac{e^{\eta_- t}-e^{\eta_+ t}}{\eta_+-\eta_-}i\xi^\tau\hat{m}_{20},\tag{21}$$

$$\hat{m}_2(\xi,t)=-i\xi\frac{8\pi\bar{\rho}+c^2|\xi|^2}{|\xi|^2}\hat{\rho}_{20}\left(\frac{e^{\eta_+t}-e^{\eta_-t}}{\eta_+-\eta_-}\right)+\frac{\xi\xi^\tau}{|\xi|^2}\left(\frac{\eta_+e^{\eta_-t}-\eta_-e^{\eta_+t}}{\eta_+-\eta_-}\right)\hat{m}_{20}+e^{-\gamma_1|\xi|^2t}\left(\boldsymbol{I}-\frac{\xi\xi^\tau}{|\xi|^2}\right)\hat{m}_{20}, \tag{22}$$

where

$$\eta_+=\frac{-(\gamma_1+\gamma_2)|\xi|^2+\sqrt{(\gamma_1+\gamma_2)^2|\xi|^4-4\left(c^2|\xi|^2+8\pi\bar{\rho}\right)}}{2},$$

$$\eta_-=\frac{-(\gamma_1+\gamma_2)|\xi|^2-\sqrt{(\gamma_1+\gamma_2)^2|\xi|^4-4\left(c^2|\xi|^2+8\pi\bar{\rho}\right)}}{2}. \tag{23}$$

When $|\xi|\leqslant\varepsilon$, and ε is small enough, we have

$$\sqrt{(\gamma_1+\gamma_2)^2|\xi|^4-4c^2|\xi|^2-32\pi\bar{\rho}}=4\sqrt{2\pi\bar{\rho}}\,i+ic^2|\xi|^2+o\left(|\xi|^2\right), \tag{24}$$

then

$$\left\|\frac{\eta_+e^{\eta_-t}}{\eta_+-\eta_-}\right\|_{L^2(|\xi|\leqslant\varepsilon)}=\left\|\left(\frac{1}{2}-\frac{1}{2}(\gamma_1+\gamma_2)|\xi|^2\cdot\frac{1}{4\sqrt{2\pi\bar{\rho}}\,i+ic^2|\xi|^2+o\left(|\xi|^2\right)}\right)e^{-\frac{1}{2}(\gamma_1+\gamma_2)|\xi|^2t}\right\|_{L^2(|\xi|\leqslant\varepsilon)}$$

$$\leqslant C\left\|e^{-\frac{1}{2}(\gamma_1+\gamma_2)|\xi|^2t}\right\|_{L^2(|\xi|\leqslant\varepsilon)}\leqslant C(1+t)^{-\frac{1}{2}}.$$

Similarly

$$\left\|\frac{\eta_+e^{\eta_-t}-\eta_-e^{\eta_+t}}{\eta_+-\eta_-}\right\|_{L^2(|\xi|\leqslant\varepsilon)}\leqslant C(1+t)^{-\frac{1}{2}}, \tag{25}$$

$$\left\|\frac{e^{\eta_+t}-e^{\eta_-t}}{\eta_+-\eta_-}i\xi^\tau\right\|_{L^2(|\xi|\leqslant\varepsilon)}\leqslant C(1+t)^{-1}, \tag{26}$$

$$\left\|\frac{\eta_+e^{\eta_-t}-\eta_-e^{\eta_+t}}{\eta_+-\eta_-}\frac{\xi\xi^\tau}{|\xi|^2}+e^{-\gamma_1|\xi|^2t}\left(\boldsymbol{I}-\frac{\xi\xi^\tau}{|\xi|^2}\right)\right\|_{L^2(|\xi|\leqslant\varepsilon)}\leqslant C(1+t)^{-\frac{1}{2}}. \tag{27}$$

From (21), (25), and (26), we have

$$\|\hat{\rho}_2\|_{L^2(|\xi|^2\leqslant\varepsilon)}\leqslant C(1+t)^{-\frac{1}{2}}\|\rho_{20}\|_{L^1}+(1+t)^{-1}\|m_{20}\|_{L^1}. \tag{28}$$

But for $\left\|\frac{8\pi\bar{\rho}\xi}{\xi^2}\cdot\frac{e^{\eta_+t}-e^{\eta_-t}}{\eta_+-\eta_-}\right\|_{L^2}$, we need a much more delicate analysis.

If $|\hat{\rho}_{20}|_{|\xi|\leqslant\varepsilon}\leqslant C_1|\xi|^{\varepsilon_1}$, for any positive ε_1, from (22) and (24), we have

$$\left\|\frac{8\pi\bar{\rho}\xi}{|\xi|^2}\cdot\frac{e^{\eta_+t}-e^{\eta_-t}}{\eta_+-\eta_-}\hat{\rho}_{20}\right\|_{L^2(|\xi|\leqslant\varepsilon)}\leqslant C\left\|\frac{1}{|\xi|}e^{-\frac{1}{2}(\gamma_1+\gamma_2)|\xi|^2t}\cdot|\xi|^{\varepsilon_1}\right\|_{L^2(|\xi|\leqslant\varepsilon)}\leqslant C\left(\int\frac{1}{|\xi|^{2-2\varepsilon_1}}e^{-(\gamma_1+\gamma_2)|\xi|^2t}|\xi|d|\xi|\right)^{\frac{1}{2}}\leqslant Ct^{-\varepsilon_1}. \tag{29}$$

From (22), (26), (27) and (29), if $|\hat{\rho}_{20}|_{|\xi|\leqslant\varepsilon}\leqslant C_1|\xi|^{\varepsilon_1}$, for any positive ε_1, we have

$$\|\hat{m}_2\|_{L^2(|\xi|^2\leqslant\varepsilon)}\leqslant Ct^{-\varepsilon_1}+C(1+t)^{-\frac{1}{2}}\|m_{20}\|_{L^1}. \tag{30}$$

when $\varepsilon\leqslant|\xi|\leqslant R, |\xi|\geqslant R$ with R large enough, using the same method as that of (11), and (12), we can also get

$$\|\hat{\rho}_2\|_{L^2(\varepsilon\leqslant|\xi|^2\leqslant R)}+\|\hat{m}_2\|_{L^2(\varepsilon\leqslant|\xi|^2\leqslant R)}\leqslant Ce^{-bt}\left(\|\rho_{20}\|_{L^1}+\|m_{20}\|_{L^1}\right), \tag{31}$$

$$\|\hat{\rho}_2\|_{L^2(|\xi|^2\geqslant R)}+\|\hat{m}_2\|_{L^2(|\xi|^2\geqslant R)}\leqslant Ce^{-bt}\left(\|\rho_{20}\|_{H^2}+\|m_{20}\|_{H^2}\right). \tag{32}$$

Together with (28), (30), (31) and (32), we have

Theorem 3 Suppose $E=\max\left(\|\rho_{10}\|_{L^1},\|m_{10}\|_{L^1},\|\rho_{10}\|_{H^2},\|m_{10}\|_{H^2}\right)$, $|\hat{\rho}_{20}|_{|\xi|\leqslant\varepsilon}\leqslant C_1|\xi|^{\varepsilon_1}$ for any positive ε_1, there exists a positive constant C such that

$$\|\hat{\rho}_2\|_{L^2} \leqslant C(1+t)^{-\frac{1}{2}}E,$$

$$\|\hat{m}_2\|_{L^2} \leqslant Ct^{-\varepsilon_1} + C(1+t)^{-\frac{1}{2}}E.$$

Theorem 4 If $|\hat{\rho}_{20}|_{|\xi|\leqslant\varepsilon} \geqslant C_1|\xi|^{\varepsilon_1}$, for any positive ε_1, we have $\|m_2\|_{L^2} \geqslant Ct^{-\varepsilon_1}$, $\|\rho_2\|_{L^2} \geqslant Ct^{-\frac{1}{2}}$.

Proof From (23) and (24), we have

$$\left\|\frac{\eta_+ e^{\eta_- t} - \eta_- e^{\eta_+ t}}{\eta_+ - \eta_-}\right\|_{L^2} \geqslant \left\|\frac{\eta_+ e^{\eta_- t} - \eta_- e^{\eta_+ t}}{\eta_+ - \eta_-}\right\|_{L^2(|\xi|\leqslant\varepsilon)} \geqslant C_1 \|\eta_+ e^{\eta_- t} - \eta_- e^{\eta_+ t}\|_{L^2(|\xi|\leqslant\varepsilon)}$$

$$\geqslant C_1 \left\| e^{-\frac{(\gamma_1+\gamma_2)|\xi|^2 t}{2}} (|\xi|^2 \sin\frac{\sqrt{(\gamma_1+\gamma_2)|\xi|^4 - 4c^2|\xi|^2 - 32\pi\bar{\rho}}}{2} t + \cos\frac{\sqrt{(\gamma_1+\gamma_2)|\xi|^4 - 4c^2|\xi|^2 - 32\pi\bar{\rho}}}{2} t) \right\|_{L^2(|\xi|\leqslant\varepsilon)}$$

$$\geqslant C \left\| e^{-\frac{(\gamma_1+\gamma_2)|\xi|^2 t}{2}} \cos\left(2\sqrt{2\pi\bar{\rho}}\cdot t + \frac{c^2|\xi|^2}{2} t\right) \right\|_{L^2(|\xi|\leqslant\varepsilon)} - C\left(\int e^{-(\gamma_1+\gamma_2)|\xi|^2 t}\left(O\left(|\xi|^4\right)t\right)^2 \mathrm{d}\xi\right)^{\frac{1}{2}} - C\left(\int |\xi|^4 e^{-(\gamma_1+\gamma_2)|\xi|^2 t} \mathrm{d}\xi\right)^{\frac{1}{2}}$$

$$= \boldsymbol{I}_1 + \boldsymbol{I}_2 + \boldsymbol{I}_3. \tag{33}$$

Fix t large enough,

$$\boldsymbol{I}_1 \geqslant \frac{1}{2} \sum_{k=\left[\frac{2\sqrt{2\pi\bar{\rho}}\,t + \frac{c^2|\xi|^2 t}{2}}{\pi} - \frac{1}{4}\right]}^{\left[\frac{2\sqrt{2\pi\bar{\rho}}\,t + \frac{c^2|\xi|^2 t}{2}}{\pi} + \frac{1}{4}\right]} \int_{2\sqrt{2\pi\bar{\rho}}\,t + \frac{c^2|\xi|^2 t}{2} \in \left[k\pi - \frac{\pi}{4}, k\pi + \frac{\pi}{4}\right]} e^{-(\gamma_1+\gamma_2)|\xi|^2 t} \mathrm{d}\xi \geqslant Ct^{-\frac{1}{2}}. \tag{34}$$

Because

$$\boldsymbol{I}_2 \leqslant Ct^{-\frac{3}{2}},\ \boldsymbol{I}_3 \leqslant Ct^{-\frac{3}{2}}, \tag{35}$$

from (33), (34) and (35), we have

$$\left\|\frac{\eta_+ e^{\eta_- t} - \eta_- e^{\eta_+ t}}{\eta_+ - \eta_-}\right\|_{L^2} \geqslant Ct^{-\frac{1}{2}}. \tag{36}$$

Similarly, we have

$$\left\|\frac{\xi}{|\xi|^2}\hat{\rho}_{20}\frac{e^{\eta_+ t} - e^{\eta_- t}}{\eta_+ - \eta_-}\right\|_{L^2} \geqslant C\left(\int_{|\xi|\geqslant\varepsilon} \frac{1}{|\xi|^{2-2\varepsilon_1}} \cdot e^{-\frac{(\gamma_1+\gamma_2)|\xi|^2 t}{2}} \cdot \sin^2 t\sqrt{32\pi\bar{\rho} + 4c^2|\xi|^2 - (\gamma_1+\gamma_2)|\xi|^4}\, \mathrm{d}\xi\right)^{\frac{1}{2}} \geqslant Ct^{-\varepsilon_1}. \tag{37}$$

Together with (21), (22), (36) and (37), we get our results.

Considering the meaning of ρ_1, m_1, ρ_2, m_2, combining Theorem 2, Theorem 3, and Theorem 4, we have the conclusion of Theorem 1 in this paper.

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二维线性等熵双极 Navier-Stokes-Poisson 方程的时间衰减

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摘要：本文考虑二维空间线性等熵双极 Navier-Stokes-Poisson 方程的柯西问题。通过对未知函数的重组，我们把原方程组转化成线性的 Navier-Stokes 和单极 Navier-Stokes-Poisson 方程组之和。通过对相应格林函数的详细分析，得到解的 L^2 衰减估计。

关键词：双极 Navier-Stokes-Poisson 方程组；格林函数；L^2 衰减

□