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The A_α-Spectral Radius and k-Extendability in Graphs

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Abstract: A graph is called k-extendable if each k-matching can be extended to a perfect matching. In this paper, we provide a sufficient condition in terms of the $\boldsymbol{A}_\alpha$-spectral radius for the k-extendability of a connected graph and characterize the corresponding extremal graphs. In addition, such an $\boldsymbol{A}_\alpha$-spectral condition of a connected balanced bipartite graph is also considered, and the corresponding extremal graphs are determined.

Key words: $\boldsymbol{A}_\alpha$-spectral radius; matching; k-extendable graph

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0 Introduction

All graphs considered are undirected, simple, and connected throughout this paper. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $|V(G)|=n$ and $|E(G)|=e(G)$ the order and the size of G, respectively. Moreover, we write $d_G(u)$ for the degree of vertex u in G, and $N_G(v)$ for the neighbor set of vertex v in G. For any $S\subseteq V(G)$, let $G[S]$ be the subgraph of G induced by S, and let $G-S$ be the subgraph induced by $V(G)\backslash S$. The set of all edges between two vertex sets X and Y is denoted by $e(X,Y)$. For any two graphs G and H, we denote by $G\cup H$ the disjoint union of G and H. The join $G\vee H$ is the graph obtained from $G\cup H$ by adding all possible edges between $V(G)$ and $V(H)$.

The adjacency matrix $\boldsymbol{A}(G)=(a_{i,j})$ of G is a $n\times n$ symmetric matrix, whose entries $a_{i,j}$ are given by $a_{i,j}=1$ if v_i and v_j are adjacent in G, and $a_{i,j}=0$ otherwise. The degree diagonal matrix of G is denoted by $\boldsymbol{D}(G)=\mathrm{diag}(d_G(v_1),d_G(v_2),\cdots,d_G(v_n))$, where $d_G(v_i)$ is the degree of v_i in G. For any real $\alpha\in[0,1]$, Nikiforov[1] defined the $\boldsymbol{A}_\alpha$-matrix of G as $\boldsymbol{A}_\alpha(G)=\alpha\boldsymbol{D}(G)+(1-\alpha)\boldsymbol{A}(G)$. The largest eigenvalue of $\boldsymbol{A}_\alpha(G)$ is called the $\boldsymbol{A}_\alpha$-spectral radius of G, and denoted by $\rho_\alpha(G)$. Obviously, $\boldsymbol{A}_0(G)=\boldsymbol{A}(G)$, $\boldsymbol{A}_1(G)=\boldsymbol{D}(G)$ and $2\boldsymbol{A}_{1/2}(G)=\boldsymbol{Q}(G)=\boldsymbol{D}(G)+\boldsymbol{A}(G)$, where $\boldsymbol{Q}(G)$ is the signless Laplacian matrix of G. Note that $\boldsymbol{A}_\alpha(G)$ is a real symmetric non-negative matrix. By the Perron-Frobenius Theorem, $\rho_\alpha(G)$ is a positive number and there exists a unique positive unit eigenvector corresponding to $\rho_\alpha(G)$, which is called the Perron vector of G.

A graph is k-extendable if each matching consisting of k edges can be extended to a perfect matching. Clearly, the

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existence of a perfect matching can be regarded as 0-extendability. The concept of k-extendability gradually evolved from the concept of elementary (that is, 1-extendable) bipartite graphs. In 1980, Plummer[2] gave the first results on k-extendable graphs for arbitrary k, he studied the properties of k-extendable graphs and proved that nearly all k-extendable graphs ($k \geqslant 2$) are $(k-1)$-extendable and $(k+1)$-connected. Then Plummer[3-4] provided the relationships between the toughness and genus of a graph and the k-extendability of the graph. Moreover, many researchers further studied the relationship between k-extendability and other graph parameters, such as minimum degree, connectivity, binding number, and independence number etc[5-10].

In the past few years, much attention has been paid to the connections of the spectral radius and the k-extendable graphs. Fan and Lin[11] investigated the adjacency spectral conditions for a graph (as well as a balanced bipartite graph) with a minimum degree δ to be k-extendable. In Ref. [12], Zhou and Zhang gave a lower bound on the signless Laplacian spectral radius to ensure that G is k-extendable. Then, Zhang and Dam[13] studied the k-extendability of graphs from a distance spectral perspective. The main goal of this paper is to study the relationship between k-extendability of graphs and the $\boldsymbol{A}_\alpha$-spectral radius.

In the following, we first provide a condition for the k-extendability of matchings in a graph in terms of the $\boldsymbol{A}_\alpha$-spectral radius.

Theorem 1 Let $k \geqslant 1$ be an integer, and let G be a connected graph of even order n with $n > 2k+4$. For $\alpha \in [0, 1)$, if

$$\rho_\alpha(G) \geqslant \rho_\alpha(K_{2k} \vee (K_{n-2k-1} \cup K_1)),$$

then G is k-extendable, unless $G \cong K_{2k} \vee (K_{n-2k-1} \cup K_1)$.

A bipartite graph is called balanced if both parts of the bipartition have equal size. Note that every bipartite graph with a perfect matching must be balanced. Hence a k-extendable bipartite graph must be balanced. Furthermore, we denote by $\mathcal{B}_{n-k,n-1}$ the bipartite graph obtained from $K_{n,n-1}$ by attaching a vertex to k vertices in n-vertex part (see Fig. 1). In Ref. [13], Zhang and Dam gave a sufficient condition in terms of the distance spectral radius for the k-extendability of a bipartite graph. Along this line, we have the result below.

Theorem 2 Let $k \geqslant 1$ be an integer, and let G be a connected balanced bipartite graph of order $2n$ with $n \geqslant k+1$. For $\alpha \in [0, 1)$, if

$$\rho_\alpha(G) \geqslant \rho_\alpha(\mathcal{B}_{n-k,n-1}),$$

then G is k-extendable, unless $G \cong \mathcal{B}_{n-k,n-1}$, see Fig. 1 for instance.

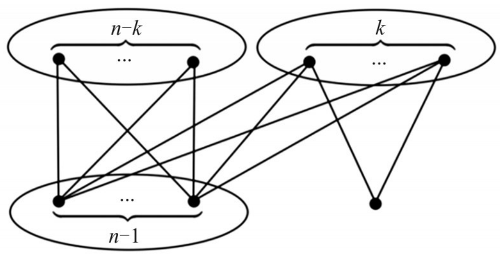

Fig. 1 The extremal graph $\mathcal{B}_{n-k,n-1}$

1 Preliminaries

In this section, we give several preliminary lemmas, which are useful to the proof of our main results.

For any $S \subseteq V(G)$, let $o(G-S)$ denote the number of odd components in $G-S$. In 1995, Chen[7] provided a necessary and sufficient condition for the existence of k-extendable graphs.

Lemma 1[7] Let $k \geqslant 1$. A graph G is k-extendable if and only if

$$o(G-S) \leqslant |S| - 2k,$$

for any $S \subseteq V(G)$ such that $G[S]$ contains k independent edges.

Lemma 2[1] If G is connected, and H is a proper subgraph of G, then $\rho_\alpha(H) < \rho_\alpha(G)$ for any $\alpha \in [0, 1)$.

Given any $X \subseteq V(G)$, let $N(X)$ be the set of all neighbors of the vertices in X. In Ref. [14], Plummer gave the necessary and sufficient conditions for bipartite graphs to be k-extendable.

Lemma 3[14] Let $k \geqslant 1$ and let G be a connected bipartite graph with parts U and W. Then the following are equivalent:

(i) G is k-extendable;

(ii) $|U| = |W|$ and for all nonempty subsets X of U, if $|X| \leqslant |U| - k$, then $|N(X)| \geqslant |X| + k$;

(iii) For all $u_1, u_2, \cdots, u_k \in U$ and $w_1, w_2, \cdots, w_k \in W$, the graph $G' = G - \{u_1, \cdots, u_k, w_1, \cdots, w_k\}$ has a perfect matching.

Finally, we need a known result mentioned as Claim 1 of the proof of Theorem 1.1 in Ref. [15].

Lemma 4[15] Let $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_q \geqslant 1$ be positive integers and $n = \sum_{i=1}^{q} n_i + s$. For $\alpha \in [0, 1)$, we have $\rho_\alpha(K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q})) \leqslant \rho_\alpha(K_s \vee (K_{n-s-q+1} \cup (q-1)K_1))$, where the equality holds if and only if $(n_1, n_2, \cdots, n_q) = (n-s-q+1, 1, \cdots, 1)$.

2 Proof of Theorem 1

In this section, we elaborate on the proof of Theorem 1, which provides a sufficient condition via the $\boldsymbol{A}_\alpha$-spectral radius of a connected graph to ensure that the graph is k-extendable.

Proof of Theorem 1 Suppose to the contrary that G is not k-extendable, we will prove that if G is not k-extendable, then $\rho_\alpha(G) \leqslant \rho_\alpha(K_{2k} \vee (K_{n-2k-1} \cup K_1))$ with equality only if $G \cong K_{2k} \vee (K_{n-2k-1} \cup K_1)$. By Lemma 1, there exists some nonempty subset $S \subseteq V(G)$, such that $|S| \geqslant 2k$ and $o(G-S) > |S| - 2k$. For convenience, let $o(G-S) = q$ and $|S| = s$. We assert that q and s have the same parity. If s is odd, then $n-s$ is an odd number since n is even, and so, the number of odd components in $G-S$ is odd, i.e., q is odd. Hence, q and s are odd. Similarly, we obtain that q is even if s is even. Thus, it concludes that $q \geqslant s-2k+2$. To promote the proof, we have the following three claims.

Claim 1 For some odd integers $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_q \geqslant 1$ and $\sum_{i=1}^{q} n_i = n-s$, G is a spanning subgraph of $K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q})$.

Proof Let $G_1, G_2, \cdots, G_q$ be the orders of the q odd components of $G-S$ with $|V(G_1)| \geqslant |V(G_2)| \geqslant \cdots \geqslant |V(G_q)|$, and let $H_1, H_2, \cdots, H_p$ be the p even components of $G-S$. Without loss of generality, we assume that all even components join to component G_1, i.e.,$G_1 \vee (H_1 \cup H_2 \cup \cdots \cup H_p)$. So, $G_1 \vee (H_1 \cup H_2 \cup \cdots \cup H_p)$ is the spanning subgraph of K_{n_1} for some integer $n_1 = |V(G_1 \vee (H_1 \cup H_2 \cup \cdots \cup H_p))|$, and $|V(G_1 \vee (H_1 \cup H_2 \cup \cdots \cup H_p))|$ is an odd integer. Let G_i be the spanning subgraph of K_{n_i} for some $2 \leqslant i \leqslant q$ and $n_i = |V(G_i)|$. Then all components of $G-S$ are odd and G is the spanning subgraph of $G[S] \vee (G-S)$, and thus, G is a spanning subgraph of $K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q})$ for some odd integers $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_q \geqslant 1$ and $\sum_{i=1}^{q} n_i = n-s$.

Furthermore, by Lemma 2 we have $\rho_\alpha(G) \leqslant \rho_\alpha(K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q}))$ with equality holds if and only if $G \cong K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q})$.

If $n_2 = n_3 = \cdots = n_q = 1$, then $K_s \vee (K_{n_1} \cup K_1 \cup \cdots \cup K_1) \cong K_s \vee (K_{n-s-(q-1)} \cup (q-1)K_1)$, say G_1.

Hence, by Lemma 4 we have that $\rho_\alpha(K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q})) \leqslant \rho_\alpha(K_s \vee (K_{n-s-q+1} \cup (q-1)K_1))$, where the equality holds if and only if $(n_1, n_2, \cdots, n_q) = (n-s-q+1, 1, \cdots, 1)$.

Note that $q \geqslant s-2k+2$. Let $G_2 \cong K_s \vee (K_{n-2s+2k-1} \cup (s-2k+1)K_1)$. We next compare the $\boldsymbol{A}_\alpha$-spectral radius of G_1 and G_2 in the following.

Claim 2 For $\alpha \in [0, 1)$, we have $\rho_\alpha(K_s \vee (K_{n-s-q+1} \cup (q-1)K_1)) \leqslant \rho_\alpha(K_s \vee (K_{n-2s+2k-1} \cup (s-2k+1)K_1))$ with equality holds if and only if $q = s-2k+2$.

Proof If $q = s-2k+2$, we have $K_s \vee (K_{n-s-q+1} \cup (q-1)K_1) \cong K_s \vee (K_{n-2s+2k-1} \cup (s-2k+1)K_1)$. Hence, $\rho_\alpha(K_s \vee (K_{n-s-q+1} \cup (q-1)K_1)) = \rho_\alpha(K_s \vee (K_{n-2s+2k-1} \cup (s-2k+1)K_1))$.

Now we consider $q > s-2k+2$. Clearly, $q \geqslant s-2k+4$ since q and s has the same parity, here it is sufficient to prove $\rho_\alpha(K_s \vee (K_{n-s-q+1} \cup (q-1)K_1)) < \rho_\alpha(K_s \vee (K_{n-2s+2k-1} \cup (s-2k+1)K_1))$ for $q \geqslant s-2k+4$. We notice that $n = s + \sum_{i=1}^{q} n_i \geqslant 2s-2k+2$, $G_1 \cong K_s \vee (K_{n-s-(q-1)} \cup (q-1)K_1)$ and $G_2 \cong K_s \vee (K_{n-2s+2k-1} \cup (s-2k+1)K_1))$.

Suppose that $\boldsymbol{X}$ is the unit Perron eigenvector of $\boldsymbol{A}_\alpha(G_1)$, and $\boldsymbol{X}_u$ denotes the entry of $\boldsymbol{X}$ corresponding to the vertex $u \in V(G_1)$. From the equitable partition $V(G_1) = V(K_s) \cup V(K_{n-s-q+1}) \cup \left(V((q-s+2k-2)K_1) \cup V\left((s-2k+1)K_1\right)\right.$, we have that $V(K_s)$, $V(K_{n-s-q+1})$ and $V((q-1)K_1)$ has the same entries in $\boldsymbol{X}$. Let $\boldsymbol{X}_u = x_1$ for $u \in V(K_s)$, $\boldsymbol{X}_v = x_2$ for $v \in V(K_{n-s-q+1})$ and $\boldsymbol{X}_w = x_3$ for $w \in V((q-1)K_1)$, respectively. For convenience, we write $\boldsymbol{X}$ as follows:

$$\boldsymbol{X} = (\underbrace{x_1, \cdots, x_1}_{s}, \underbrace{x_2, \cdots, x_2}_{n-s-q+1}, \underbrace{x_3, \cdots, x_3}_{q-s+2k-2}, \underbrace{x_3, \cdots, x_3}_{s-2k+1})^{\mathrm{T}}.$$

For the graph G_2, we partition $V(G_2)$ as $V(K_s) \cup V(K_{n-s-q+1}) \cup V(K_{q-s+2k-2}) \cup V((s-2k+1)K_1)$. Thus, $\boldsymbol{A}_\alpha(G_2) - \boldsymbol{A}_\alpha(G_1)$ has the form of a block matrix below:

$$\begin{pmatrix} \boldsymbol{O}_{n_1\times n_1} & \boldsymbol{O}_{n_1\times n_2} & \boldsymbol{O}_{n_1\times n_3} & \boldsymbol{O}_{n_1\times n_4} \\ \boldsymbol{O}_{n_2\times n_1} & \alpha(q-s+2k-2)\boldsymbol{I}_{n_2\times n_2} & (1-\alpha)\boldsymbol{I}_{n_2\times n_3} & \boldsymbol{O}_{n_2\times n_4} \\ \boldsymbol{O}_{n_3\times n_1} & (1-\alpha)\boldsymbol{I}_{n_3\times n_2} & \alpha(n-2s+2k-2)\boldsymbol{I}_{n_3\times n_3} + (1-\alpha)(\boldsymbol{J}-\boldsymbol{I})_{n_3\times n_3} & \boldsymbol{O}_{n_3\times n_4} \\ \boldsymbol{O}_{n_4\times n_1} & \boldsymbol{O}_{n_4\times n_2} & \boldsymbol{O}_{n_4\times n_3} & \boldsymbol{O}_{n_4\times n_4} \end{pmatrix},$$

where $n_1 = s, n_2 = n-s-q+1, n_3 = q-s+2k-2$ and $n_4 = s-2k+1$, meanwhile, $\boldsymbol{J}$ is an all-ones matrix and $\boldsymbol{I}$ is an identity matrix.

Furthermore, by Rayleigh's principle, we have

$$\begin{aligned} &\rho_\alpha(G_2) - \rho_\alpha(G_1) \\ &\geqslant \boldsymbol{X}^{\mathrm{T}}(\boldsymbol{A}_\alpha(G_2) - \boldsymbol{A}_\alpha(G_1))\boldsymbol{X} \\ &= \alpha(q-s+2k-2)((n-s-q+1)x_2^2 + (n-2s+2k-2)x_3^2) + (1-\alpha)(q-s+2k-2) \\ &\quad \times (2(n-s-q+1)x_2x_3 + (q-s+2k-3)x_3^2) \\ &= (q-s+2k-2)(\alpha((n-s-q+1)x_2^2 + (n-2s+2k-2)x_3^2) + (1-\alpha)(2(n-s-q+1) \\ &\quad x_2x_3 + (q-s+2k-3)x_3^2)) \\ &\geqslant 2(\alpha((n-s-q+1)x_2^2 + (n-2s+2k-2)x_3^2) + (1-\alpha)((n-s-q+1)2x_2x_3 + x_3^2)) \\ &\quad (\text{since } q \geqslant s-2k+4) \\ &\geqslant 2(\alpha(n-s-q+1)x_2^2 + (1-\alpha)((n-s-q+1)2x_2x_3 + x_3^2))\ (\text{since } n \geqslant 2s-2k+2) \\ &> 0. \end{aligned}$$

Therefore, $\rho_\alpha(K_s \vee (K_{n-s-q+1} \cup (q-1)K_1)) < \rho_\alpha(K_s \vee (K_{n-2s+2k-1} \cup (s-2k+1)K_1))$.

Recall that $s \geqslant 2k$. Let $G_3 \cong K_{2k} \vee (K_{n-2k-1} \cup K_1)$. Next, we compare the $\boldsymbol{A}_\alpha$-spectral radius of G_2 and G_3 and give the following claim.

Claim 3 Let $k \geqslant 1$ and $n > 2k+4$. For $\alpha \in [0, 1)$, we have $\rho_\alpha(K_s \vee (K_{n-2s+2k-1} \cup (s-2k+1)K_1)) \leqslant \rho_\alpha(K_{2k} \vee (K_{n-2k-1} \cup K_1))$ with equality holds if and only if $s = 2k$.

Proof If $s = 2k$, we have $K_s \vee (K_{n-2s+2k-1} \cup (s-2k+1)K_1) \cong K_{2k} \vee (K_{n-2k-1} \cup K_1)$. Hence, $\rho_\alpha(K_s \vee (K_{n-2s+2k-1} \cup (s-2k+1)K_1)) = \rho_\alpha(K_{2k} \vee (K_{n-2k-1} \cup K_1))$.

If $s \geqslant 2k+1$, we should verify that $\rho_\alpha(K_s \vee (K_{n-2s+2k-1} \cup (s-2k+1)K_1)) < \rho_\alpha(K_{2k} \vee (K_{n-2k-1} \cup K_1))$. Since $n \geqslant 2s-2k+2$ we have $s \leqslant \dfrac{n}{2} + k - 1$. Thus, it concludes that $2k+1 \leqslant s \leqslant \dfrac{n}{2} + k - 1$. Note that $G_2 \cong K_s \vee (K_{n-2s+2k-1} \cup (s-2k+1)K_1)$ and $G_3 \cong K_{2k} \vee (K_{n-2k-1} \cup K_1)$.

Let $\boldsymbol{Y}$ be the unit Perron eigenvector of $\boldsymbol{A}_\alpha(G_3)$, and $\boldsymbol{Y}_v$ be the entry of $\boldsymbol{Y}$ corresponding to the vertex $v \in V(G_3)$. Since $V(G_3) = V(K_{2k}) \cup (V(K_{s-2k}) \cup V(K_{n-2s+2k-1}) \cup V(K_{s-2k})) \cup V(K_1)$ is an equitable partition of G_3, $\boldsymbol{Y}$ takes the same value on the vertices of $V(K_{2k})$ (respectively $V(K_{n-2k-1})$ and $V(K_1)$). Let $\boldsymbol{Y}_u = y_1$ for $u \in V(K_{2k})$, $\boldsymbol{Y}_v = y_2$ for $v \in V(K_{n-2k-1})$ and $\boldsymbol{Y}_w = y_3$ for $w \in V(K_1)$, then $\boldsymbol{Y}$ can be written below:

$$\boldsymbol{Y} = (\underbrace{y_1, \cdots, y_1}_{2k}, \underbrace{y_2, \cdots, y_2}_{s-2k}, \underbrace{y_2, \cdots, y_2}_{n-2s+2k-1}, \underbrace{y_2, \cdots, y_2}_{s-2k}, y_3)^{\mathrm{T}}.$$

Let $V(G_2)=V(K_s)\cup V(K_{n-s-q+1})\cup V(K_{q-s+2k-2})\cup V((s-2k+1)K_1)$. Then $\boldsymbol{A}_\alpha(G_3)-\boldsymbol{A}_\alpha(G_2)$ can be partitioned as

$$\begin{pmatrix} \boldsymbol{O}_{m_1\times m_1} & \boldsymbol{O}_{m_1\times m_2} & \boldsymbol{O}_{m_1\times m_3} & \boldsymbol{O}_{m_1\times m_2} & \boldsymbol{O}_{m_1\times 1} \\ \boldsymbol{O}_{m_2\times m_1} & -\alpha\boldsymbol{I}_{m_2\times m_2} & \boldsymbol{O}_{m_2\times m_3} & \boldsymbol{O}_{m_2\times m_2} & -(1-\alpha)\boldsymbol{I}_{m_2\times 1} \\ \boldsymbol{O}_{m_3\times m_1} & \boldsymbol{O}_{m_3\times m_2} & \alpha(s-2k)\boldsymbol{I}_{m_3\times m_3} & (1-\alpha)\boldsymbol{I}_{m_3\times m_2} & \boldsymbol{O}_{m_3\times 1} \\ \boldsymbol{O}_{m_2\times m_1} & \boldsymbol{O}_{m_2\times m_2} & (1-\alpha)\boldsymbol{I}_{m_2\times m_3} & \alpha(n-s-2)\boldsymbol{I}_{m_2\times m_2}+(1-\alpha)(\boldsymbol{J}-\boldsymbol{I})_{m_2\times m_2} & \boldsymbol{O}_{m_2\times 1} \\ \boldsymbol{O}_{1\times m_1} & -(1-\alpha)\boldsymbol{I}_{1\times m_2} & \boldsymbol{O}_{1\times m_3} & \boldsymbol{O}_{1\times m_2} & \alpha(2k-s)\boldsymbol{I}_{1\times 1} \end{pmatrix}$$

where $m_1=2k$, $m_2=s-2k$ and $m_3=n-2s+2k-1$.

Thus, by $\boldsymbol{A}_\alpha(G_3)\boldsymbol{Y}=\rho_\alpha(G_3)\boldsymbol{Y}$, we can get that

$$\rho_\alpha(G_3)y_2=\alpha(n-2)y_2+(1-\alpha)(2ky_1+(n-2k-2)y_2), \tag{1}$$

$$\rho_\alpha(G_3)y_3=\alpha(2k)y_3+(1-\alpha)2ky_1. \tag{2}$$

Hence, it follows from Eqs. (1) and (2) that

$$(\rho_\alpha(G_3)-2\alpha k-(n-2k-2))y_2=(\rho_\alpha(G_3)-2\alpha k)y_3,$$

which implies $y_2>y_3$.

According to the Rayleigh quotient, we derive that

$$\begin{aligned}
&\rho_\alpha(G_3)-\rho_\alpha(G_2)\\
&\geqslant \boldsymbol{Y}^{\mathrm{T}}(\boldsymbol{A}_\alpha(G_3)-\boldsymbol{A}_\alpha(G_2))\boldsymbol{Y}\\
&=\alpha((s-2k)(2n-3s+2k-4)y_2^2+(2k-s)y_3^2)+(1-\alpha)((s-2k)(2n-3s+2k-3)y_2^2-2(s-2k)y_2y_3)\\
&=(s-2k)(\alpha((2n-3s+2k-4)y_2^2-y_3^2)+(1-\alpha)((2n-3s+2k-3)y_2^2-2y_2y_3))\\
&>(s-2k)(\alpha((2n-3s+2k-4)y_2^2-y_2^2)+(1-\alpha)((2n-3s+2k-3)y_2^2-2y_2^2))(\text{since } y_2>y_3)\\
&=(s-2k)(\alpha(2n-3s+2k-5)y_2^2+(1-\alpha)(2n-3s+2k-5)y_2^2)\\
&=(s-2k)(2n-3s+2k-5)y_2^2\\
&>0\ (\text{since } 2k\leqslant s\leqslant \frac{n}{2}+k-1 \text{ and } n>2k+4).
\end{aligned}$$

Thus, $\rho_\alpha(K_s\vee(K_{n-2s+2k-1}\cup(s-2k+1)K_1))<\rho_\alpha(K_{2k}\vee(K_{n-2k-1}\cup K_1))$.

Moreover, we prove that $K_{2k}\vee(K_{n-2k-1}\cup K_1)$ is not k-extendable. Set $V_1=V(K_{2k})$, $V_2=V(K_{n-2k-1})$ and $V_3=V(K_1)$, we take a nonempty subset $S=V_1$, then $o(G-S)=2>|S|-2k$. Hence, Lemma 1 deduces that $K_{2k}\vee(K_{n-2k-1}\cup K_1)$ is not k-extendable. Therefore, the proof is completed.

3 Proof of Theorem 2

In this section, we will present a comprehensive proof of Theorem 2.

Proof of Theorem 2 Suppose, by a contradiction, that G is not k-extendable. We will prove that if G is connected balanced bipartite but not k-extendable, then $\rho_\alpha(G)\leqslant\rho_\alpha(\mathcal{B}_{n-k,n-1})$ with equality holds if and only if $G\cong\mathcal{B}_{n-k,n-1}$.

Let $G=G[U,W]$ be a connected balanced bipartite graph, where $|U|=|W|=n$. Since G is not k-extendable, by Lemma 3, there exists some nonempty subset $X\subseteq U$ such that $|N(X)|\leqslant|X|+k-1$ and $|X|\leqslant n-k$. Let $\mathcal{B}_{|X|,|N(X)|}$ be the connected balanced bipartite graph be obtained from G by joining X and $N(X)$ as well as joining $U-X$ and $W-N(X)$, and by adding all possible edges between $U-X$ and $N(X)$, i.e., $\mathcal{B}_{|X|,|N(X)|}\cong K_{|X|,|N(X)|}\cup K_{n-|X|,n-|N(X)|}+e(N(X),U-X)$.

For convenience, we may set $|X|=s$. It is clear then that G is a spanning subgraph of $\mathcal{B}_{s,s+k-1}$, and

$$\mathcal{B}_{s,s+k-1}\cong K_{s,s+k-1}\cup K_{n-s,n-s-k+1}+e(N(X),U-X).$$

By Lemma 2, we can deduce that

$$\rho_\alpha(G)\leqslant\rho_\alpha(\mathcal{B}_{s,s+k-1})$$

with equality holds if and only if $G\cong\mathcal{B}_{s,s+k-1}$.

If $s=1$ or $s=n-k$, then $\mathcal{B}_{1,k}=\mathcal{B}_{n-k,n-1}$, which is the claimed extremal graph. By the symmetry of $\mathcal{B}_{s,s+k-1}=\mathcal{B}_{n-k-s+1,n-s}$, we need to prove $\rho_\alpha(\mathcal{B}_{s,s+k-1})<\rho_\alpha(\mathcal{B}_{n-k,n-1})$ for $2\leqslant s\leqslant\frac{1}{2}(n-k+1)$.

Let $\boldsymbol{Z}$ be the unit Perron eigenvector of $\boldsymbol{A}_\alpha(\mathcal{B}_{n-k,n-1})$, and $\boldsymbol{Z}_u$ denotes the entry of $\boldsymbol{Z}$ corresponding to the vertex

$u \in V(\mathcal{B}_{n-k,n-1})$. It is easy to see that $V(\mathcal{B}_{n-k,n-1})=(V(sK_1)\cup V((n-k-s)K_1))\cup V(kK_1)\cup(V((s+k-1)K_1)\cup V((n-s-k)K_1))\cup V(K_1)$ is an equitable partition of $\mathcal{B}_{n-k,n-1}$, and so, all vertices of $V((n-k)K_1)$ (respectively $V(kK_1)$, $V((n-1)K_1)$ and $V(K_1)$) have the same entries in $\boldsymbol{Z}$. Then we may set $\boldsymbol{Z}_u=z_1$ for $u \in V((n-k)K_1)$, $\boldsymbol{Z}_v=z_2$ for $v \in V(kK_1)$, $\boldsymbol{Z}_w=z_3$ for $w \in V((n-1)K_1)$ and $\boldsymbol{Z}_t=z_4$ for $t \in V(K_1)$, respectively. So, we write

$$\boldsymbol{Z}=(\underbrace{z_1,\cdots,z_1}_{s},\underbrace{z_1,\cdots,z_1}_{n-k-s},\underbrace{z_2,\cdots,z_2}_{k},\underbrace{z_3,\cdots,z_3}_{s+k-1},\underbrace{z_3,\cdots,z_3}_{n-s-k},z_4)^{\mathrm{T}}.$$

Let $V(\mathcal{B}_{s,s+k-1})=V(sK_1)\cup V((n-k-s)K_1)\cup V(kK_1)\cup V((s+k-1)K_1)\cup V((n-s-k)K_1)\cup V(K_1)$. Then one can partition $\boldsymbol{A}_\alpha(\mathcal{B}_{n-k,n-1})-\boldsymbol{A}_\alpha(\mathcal{B}_{s,s+k-1})$ as follows:

$$\begin{pmatrix} \alpha(n-s-k)\boldsymbol{I}_{n_1\times n_1} & \boldsymbol{O}_{n_1\times n_2} & \boldsymbol{O}_{n_1\times n_3} & \boldsymbol{O}_{n_1\times n_4} & (1-\alpha)\boldsymbol{I}_{n_1\times n_2} & \boldsymbol{O}_{n_1\times 1} \\ \boldsymbol{O}_{n_2\times n_1} & -\alpha\boldsymbol{I}_{n_2\times n_2} & \boldsymbol{O}_{n_2\times n_3} & \boldsymbol{O}_{n_2\times n_4} & \boldsymbol{O}_{n_2\times n_2} & -(1-\alpha)\boldsymbol{I}_{n_2\times 1} \\ \boldsymbol{O}_{n_3\times n_1} & \boldsymbol{O}_{n_3\times n_2} & \boldsymbol{O}_{n_3\times n_3} & \boldsymbol{O}_{n_3\times n_4} & \boldsymbol{O}_{n_3\times n_2} & \boldsymbol{O}_{n_3\times 1} \\ \boldsymbol{O}_{n_4\times n_1} & \boldsymbol{O}_{n_4\times n_2} & \boldsymbol{O}_{n_4\times n_3} & \boldsymbol{O}_{n_4\times n_4} & \boldsymbol{O}_{n_4\times n_2} & \boldsymbol{O}_{n_4\times 1} \\ (1-\alpha)\boldsymbol{I}_{n_2\times n_1} & \boldsymbol{O}_{n_2\times n_2} & \boldsymbol{O}_{n_2\times n_3} & \boldsymbol{O}_{n_2\times n_4} & \alpha s\boldsymbol{I}_{n_2\times n_2} & \boldsymbol{O}_{n_2\times 1} \\ \boldsymbol{O}_{1\times n_1} & -(1-\alpha)\boldsymbol{I}_{1\times n_2} & \boldsymbol{O}_{1\times n_3} & \boldsymbol{O}_{1\times n_4} & \boldsymbol{O}_{1\times n_2} & \alpha(k-n+s)\boldsymbol{I}_{1\times 1} \end{pmatrix}$$

where $n_1=s$, $n_2=n-k-s$, $n_3=k$ and $n_4=s+k-1$.

Thus, by $\boldsymbol{A}_\alpha(\mathcal{B}_{n-k,n-1})\boldsymbol{Z}=\rho_\alpha(\mathcal{B}_{n-k,n-1})\boldsymbol{Z}$, we can get

$$\rho_\alpha(\mathcal{B}_{n-k,n-1})z_1=\alpha(n-1)z_1+(1-\alpha)(n-1)z_3, \tag{3}$$

$$\rho_\alpha(\mathcal{B}_{n-k,n-1})z_3=\alpha nz_3+(1-\alpha)((n-k)z_1+kz_2), \tag{4}$$

$$\rho_\alpha(\mathcal{B}_{n-k,n-1})z_4=\alpha kz_4+(1-\alpha)(kz_2). \tag{5}$$

Now, from Eq. (3) we have

$$z_1=\frac{(1-\alpha)(n-1)}{\rho_\alpha(\mathcal{B}_{n-k,n-1})-\alpha(n-1)}z_3. \tag{6}$$

Moreover, by Eqs. (4) and (5) we deduce that

$$(\rho_\alpha(\mathcal{B}_{n-k,n-1})-\alpha n)z_3-(1-\alpha)(n-k)z_1=(\rho_\alpha(\mathcal{B}_{n-k,n-1})-\alpha k)z_4. \tag{7}$$

Consequently, combining with Eqs. (6) and (7) we have

$$(\rho_\alpha(\mathcal{B}_{n-k,n-1})-\alpha n-\frac{(1-\alpha)^2(n-k)(n-1)}{\rho_\alpha(\mathcal{B}_{n-k,n-1})-\alpha(n-1)})z_3=(\rho_\alpha(\mathcal{B}_{n-k,n-1})-\alpha k)z_4,$$

which implies that $z_3>z_4$.

Moreover, by Rayleigh's principle, we can get

$$\begin{aligned} &\rho_\alpha(\mathcal{B}_{n-k,n-1})-\rho_\alpha(\mathcal{B}_{s,s+k-1}) \\ &\geqslant \boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{A}_\alpha(\mathcal{B}_{n-k,n-1})-\boldsymbol{A}_\alpha(\mathcal{B}_{s,s+k-1}))\boldsymbol{Z} \\ &=\alpha((n-s-k)(s-1)z_1^2+s(n-s-k)z_3^2+(k-n+s)z_4^2)+(1-\alpha)(2s(n-s-k)z_1z_3-2(n-s-k)z_1z_4) \\ &=(n-s-k)(\alpha((s-1)z_1^2+sz_3^2-z_4^2)+(1-\alpha)(2sz_1z_3-2z_1z_4)) \\ &>(n-s-k)(\alpha((s-1)z_1^2+sz_3^2-z_3^2)+(1-\alpha)(2sz_1z_3-2z_1z_3))\,(\text{since } z_3>z_4) \\ &=(n-s-k)(\alpha((s-1)z_1^2+(s-1)z_3^2)+(1-\alpha)(2(s-1)z_1z_3)) \\ &>0\,(\text{since } 2\leqslant s\leqslant \frac{1}{2}(n-k+1)). \end{aligned}$$

Hence, it deduces that $\rho_\alpha(\mathcal{B}_{n-k,n-1})>\rho_\alpha(\mathcal{B}_{s,s+k-1})$. Note that the graph $\mathcal{B}_{n-k,n-1}$ is not k-extendable. Therefore, this completes the proof of Theorem 2.

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图的$\boldsymbol{A}_\alpha$-谱半径与k-可扩展性

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摘要：若一个图的任意k-匹配都可扩展为该图的完美匹配，则称此图为k-可扩展图。本文给出了连通图具有k-可扩展性的一个$\boldsymbol{A}_\alpha$-谱条件，并刻画了相应的极图。此外，还考虑了连通平衡二部图具有k-可扩展性的一个$\boldsymbol{A}_\alpha$-谱条件及其相应的极图。

关键词：$\boldsymbol{A}_\alpha$-谱半径；匹配；k-可扩展图

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