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Absolutely Clean N -Complexes

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Abstract: The notion of absolutely clean N -complexes is studied. We show that an N -complex X is absolutely clean if and only if X is N -exact and $Z_n^i(X)$ is an absolutely clean module for each $n \in \mathbb{Z}$ and $i = 1, 2, \dots, N$. In particular, we prove that a bounded above N -complex X is absolutely clean if and only if X_n is an absolutely clean module for each $n \in \mathbb{Z}$. We also show that under certain hypotheses, an N -complex X is Gorenstein AC-injective if and only if $Z_n^t(X)$ is a Gorenstein AC-injective module for each $n \in \mathbb{Z}$ and $t = 1, 2, \dots, N$.

Key words: absolutely clean module; absolutely clean N -complex; Gorenstein AC-injective N -complex

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0 Introduction

A module M is called type FP_∞ if M has a projective resolution by finitely generated projective modules. This concept is a generalization of finitely generated modules and finitely presented modules, which is studied by Bravo *et al*^[1]. Also a module N is called FP_∞ -injective or absolutely clean if $\text{Ext}_R^1(M, N) = 0$ for all modules M of type FP_∞ , these modules have properties that injective modules only have over Noetherian rings. Specifically, if R is left Noetherian, the modules of type FP_∞ are precisely the finitely generated modules.

The category of complexes plays an important role in the theory of homological algebra, and many results of the category of modules have been generalized to the category of complexes. As we know, in the category of complexes, the relationship between complexes and its level modules and cycle modules is an important research topic. For example, a complex is injective if and only if it is exact and each cycle module is injective. A

complex X is absolutely clean if and only if X is exact and each $Z_n^t(X)$ is an absolutely clean module for each $n \in \mathbb{Z}$ and $t = 1, 2, \dots, N$. A complex X is Gorenstein AC-injective if and only if X_n is a Gorenstein AC-injective module and $\text{Hom}_R(A, -)$ is exact for any absolutely clean complex A ^[2].

The notion of N -complexes was introduced by Mayer in his study of simplicial complexes^[3-4], that is, it satisfies differentials $d^N = 0$. The category of N -complexes is consistent with the category of complexes whenever $N = 2$. In 1996, Kapranov^[5], Dubois-Violette and Kerner^[6] gave an abstract framework of homological theory of N -complexes. Since then, the N -complexes have been concerned by many authors, for example Refs. [7-11]. Recently, Lu^[12] introduced the concept of FP -injective N -complexes, which has shown that an N -complex X is FP -injective if and only if X is N -exact and $Z_n^t(X)$ is an FP -injective module for each $n \in \mathbb{Z}$ and $t = 1, 2, \dots, N$.

In present paper, we establish relationships between

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the absolutely clean N -complex and its level modules and cycle modules. And from this, we prove that under certain hypotheses, an N -complex X is Gorenstein AC-injective if and only if $Z_n^t(X)$ is a Gorenstein AC-injective module for each $n \in \mathbb{Z}$ and $t = 1, 2, \dots, N$.

More precisely, our results can be stated as follows:

Theorem 1 Let X be an N -complex. Then X is absolutely clean if and only if X is N -exact and $Z_n^i(X)$ is an absolutely clean module for each $n \in \mathbb{Z}$ and $i = 1, 2, \dots, N$.

As an application of Theorem 1, the following result is established.

Proposition 1 Let X be a bounded above N -complex. Then X is absolutely clean if and only if X is N -exact and X_n is an absolutely clean module for $n \in \mathbb{Z}$.

It is well known that an N -complex X is Gorenstein AC-injective if and only if X_n is a Gorenstein AC-injective module and $\text{Hom}_R(A, X)$ is N -exact for any absolutely clean N -complex A ^[9]. As another application of Theorem 1, we also obtain the following result, which extends the above classical result to the Gorenstein AC-injective N -complexes.

Proposition 2 Let X be an N -exact N -complex with $\text{Hom}_R(D_n^1(M), X)$ N -exact for any absolutely clean module M . Then X is a Gorenstein AC-injective N -complex if and only if $Z_n^t(X)$ is a Gorenstein AC-injective module for each $n \in \mathbb{Z}$ and $t = 1, 2, \dots, N$.

1 Preliminaries

Throughout this paper, unless stated otherwise, R denotes an associative ring with an identity and by the term "module" we always mean a left R -module and use $R\text{-Mod}$ to denote the category of left R -modules.

This section is devoted to recalling some notions and basic consequences for use throughout this paper. For terminology we shall follow Refs. [7] and [10] when working with N -complexes.

By an N -complex X ($N \geq 2$) we mean a sequence of left R -modules

$$\cdots \xrightarrow{d_{n+2}^X} X_{n+1} \xrightarrow{d_{n+1}^X} X_n \xrightarrow{d_n^X} X_{n-1} \xrightarrow{d_{n-1}^X} \cdots$$

satisfying $d^n = d_{n+1}^X d_{n+2}^X \cdots d_{n+N}^X = 0$ for any $n \in \mathbb{Z}$. That is, composing any N -consecutive morphisms gives 0. So a 2-complex is a chain complex in the usual sense. A chain map or simply map $f: X \rightarrow Y$ of N -complexes is a collection of morphisms $f_n: X_n \rightarrow Y_n$ making all the rectangles commute. In this way, we get a category of N -complexes

of left R -modules, denoted by $\mathcal{C}_N(R)$, whose objects are N -complexes and whose morphisms are chain maps. This is an abelian category having enough projectives and injectives. Let C and D be N -complexes. We use $\text{Hom}_{\mathcal{C}_N(R)}(C, D)$ to denote the Abelian group of morphisms from C and D and $\text{Ext}_{\mathcal{C}_N(R)}^i(C, D)$ for $i \geq 0$ to denote the groups we get from the right derived functor of Hom .

For an N -complex X , there are $N-1$ choices for homology. Indeed for $t = 1, 2, \dots, N$, we define $Z_n^t(X) = \text{Ker}(d_{n-(t-1)} \cdots d_{n-1} d_n)$ and $B_n^t(X) = \text{Im}(d_{n+1} d_{n+2} \cdots d_{n+t})$. In particular, we have $Z_n^1(X) = \text{Ker} d_n$, $Z_n^N(X) = X_n$ and $B_n^1(X) = \text{Im} d_{n+1}$, $B_n^N(X) = 0$. Finally, we define $H_n^t(X) = Z_n^t(X)/B_n^{N-t}(X)$ the amplitude homology objects of X for all t . We say X is N -exact, or just exact, if $H_n^t(X) = 0$ for all n and t .

An N -complex X is called bounded, if there is only a finite number n such that $X_n \neq 0$; if $X_n = 0$ when n is sufficiently large, which is called bounded above. Similarly, we can define lower bounded N -complexes. As we know, an N -complex X is called finitely generated if X is bounded and X_n is a finitely generated module for each $n \in \mathbb{Z}$.

Unless stated otherwise, in the following complexes will always denote 2-complexes. Given a module A , we define N -complexes $D_n^t(A)$ for $t = 1, 2, \dots, N$, as follows: $D_n^t(A)$ consists of A in degrees $n, n-1, \dots, n-(t-1)$, all joined by identity morphisms, and 0 in every other degree. Let $\{M_n | n \in \mathbb{Z}\}$ be objects of modules. Then $(\prod_{n \in \mathbb{Z}} D_n^N(M_n))_k = (\prod_{n \in \mathbb{Z}} D_n^N(M_n))_k = M_{k+N-1} \oplus \cdots \oplus M_k$ for all k . Therefore,

$$\prod_{n \in \mathbb{Z}} D_n^N(M_n) = \prod_{n \in \mathbb{Z}} D_n^N(M_n).$$

Two chain maps $f, g: X \rightarrow Y$ are called chain homotopic, or simply homotopic if there exists a collection of morphisms $\{s_n: X_n \rightarrow Y_{n+N-1}\}$ such that

$$\begin{aligned} g_n - f_n &= d^{N-1} s_n + d^{N-2} s_{n-1} d + \cdots + s_{n-(N-1)} d^{N-1} \\ &= \sum_{i=0}^{N-1} d^{(N-1)-i} s_{n-i} d^i, \quad \forall n. \end{aligned} \quad (1)$$

If f and g are homotopic, then we write $f \sim g$. We call a chain map f null homotopic if $f \sim 0$. There exists an additive category $\mathcal{K}_N(R)$, called the homotopy category of N -complexes, whose objects are the same as those of $\mathcal{C}_N(R)$ and whose Hom sets are the equivalence classes of Hom sets in $\mathcal{C}_N(R)$. An isomorphism in $\mathcal{K}_N(R)$ is called a homotopy equivalence. Let $\mathcal{S}_N(R)$ be the collection of sequences $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ of morphisms in

$\mathcal{C}_N(R)$ such that $0 \rightarrow X_i \rightarrow Y_i \rightarrow Z_i \rightarrow 0$ is split exact in $\mathcal{K}_N(R)$ for any integer i .

Following Ref. [8], let $X \in \mathcal{C}_N(R)$, we have morphisms $\rho_n^{X_{n+N-1}}: D_{n+N-1}^N(X_{n+N-1}) \rightarrow X$ and $\lambda_n^{X_n}: X \rightarrow D_{n+N-1}^N(X_n)$. Set $\rho^X: \bigoplus_{n \in \mathbb{Z}} D_{n+N-1}^N(X_{n+N-1}) \rightarrow X$ and $\lambda^X: X \rightarrow \bigoplus_{n \in \mathbb{Z}} D_{n+N-1}^N(X_n)$. Then we have the following exact sequences in $\mathcal{S}_N(R)$,

$$0 \rightarrow \text{Ker} \rho^X \xrightarrow{\varepsilon^X} \bigoplus_{n \in \mathbb{Z}} D_n^N(X_n) \xrightarrow{\rho^X} X \rightarrow 0 \quad (2)$$

and

$$0 \rightarrow X \xrightarrow{\lambda^X} \bigoplus_{n \in \mathbb{Z}} D_{n+N-1}^N(X_n) \xrightarrow{\eta^X} \text{Coker} \lambda^X \rightarrow 0. \quad (3)$$

Now we define functors $\Sigma, \Sigma^{-1}: \mathcal{C}_N(R) \rightarrow \mathcal{C}_N(R)$ by $\Sigma^{-1}X = \text{Ker} \rho^X$ and $\Sigma X = \text{Coker} \lambda^X$ in the exact sequences above. Then Σ and Σ^{-1} induce the suspension functor and its quasi-inverse of the triangulated category $\mathcal{K}_N(R)$. On the other hand, we define the shift functor $\Theta: \mathcal{C}_N(R) \rightarrow \mathcal{C}_N(R)$ by $\Theta(X)_i = X_{i+1}$ and $d_i^{\Theta(X)} = d_{i+1}^X$ for $X = (X_i, d_i^X) \in \mathcal{C}_N(R)$. The N -complex $\Theta(\Theta X)$ is denoted as $\Theta^2 X$ and inductively we define $\Theta^n X$ for all $n \in \mathbb{Z}$. This induces the shift functor $\Theta: \mathcal{K}_N(R) \rightarrow \mathcal{K}_N(R)$ which is a triangle functor. Unlike classical case, Σ does not coincide with Θ .

Following Refs. [8] or [10], for any N -complex X , ΣX and $\Sigma^{-1}X$ are given by the following explicit description,

$$\begin{aligned} (\Sigma X)_n &= X_{n-1} \oplus X_{n-2} \oplus \cdots \oplus X_{n-(N-1)}, \\ d_n^{\Sigma X} &= \begin{pmatrix} -d & 1 & 0 & \cdots & 0 & 0 & 0 \\ -d^2 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -d^{N-3} & 0 & 0 & \cdots & 0 & 1 & 0 \\ -d^{N-2} & 0 & 0 & \cdots & 0 & 0 & 1 \\ -d^{N-1} & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, \\ (\Sigma^{-1}X)_n &= X_{n+N-1} \oplus \cdots \oplus X_{n+2} \oplus X_{n+1}, \\ d_n^{\Sigma^{-1}X} &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -d^{N-1} & -d^{N-2} & -d^{N-3} & \cdots & -d^3 & -d^2 & -d \end{pmatrix}. \end{aligned}$$

2 Main Results

In this section, we investigate the concept of absolutely clean N -complexes and give some equivalence characterizations of absolutely clean N -complexes. In particular, we prove that an N -complex X is absolutely clean if and only if X is N -exact and $Z_n^i(X)$ is an abso-

lutely clean module for each $n \in \mathbb{Z}$ and $i = 1, 2, \dots, N$.

Definition 1 An N -complex C is of type FP_∞ if C has a projective resolution $\cdots \rightarrow P^2 \rightarrow P^1 \rightarrow P^0 \rightarrow C \rightarrow 0$ by finitely generated projective N -complexes P^i for $i \geq 0$.

Proposition 3 An N -complex C is of type FP_∞ if and only if it is bounded and each C_n is a module of type FP_∞ .

Proof Sufficiency. Let C be of type FP_∞ . Then it must be finitely generated, so it is bounded. We also see that each N -complex P^i must consist of finitely generated projective modules in each degree by the definition of N -complex of type FP_∞ . So we get that each C_n is a module of type FP_∞ .

Necessity. Suppose C is bounded and each C_n is a module of type FP_∞ , we can construct a surjection $P^0 \xrightarrow{f} C$ where P^0 is a finitely generated projective N -complex. Set $K_0 = \text{Ker} f$ and note that it also must be bounded. Since each C_n must also be finitely presented, it follows that each K_n is finitely generated. Thus K_0 is finitely generated, and we can again construct a surjection $P^1 \xrightarrow{f_1} K_0$ where P^1 is a finitely generated projective N -complex. Set $K_1 = \text{Ker} f_1$ and note that K_1 must be bounded. Since each C_n must be of type FP_∞ , it follows that K_1 must also be finitely generated N -complex. Continuing this way, we can construct a projective resolution

$$\cdots \rightarrow P^2 \rightarrow P^1 \rightarrow P^0 \rightarrow C \rightarrow 0 \quad (4)$$

where each P^i is a finitely generated projective N -complex.

Remark 1 1) If M is a module of type FP_∞ , then $D_n^t(M)$ is an N -complex of type FP_∞ for each $n \in \mathbb{Z}$ and $t = 1, 2, \dots, N$.

2) An N -complex of type FP_∞ is finitely generated.

Definition 2 An N -complex X is said to be absolutely clean if $\text{Ext}_{\mathcal{C}_N(R)}^1(P, X) = 0$ for all N -complexes P of type FP_∞ .

Remark 2 1) If $N = 2$, then absolutely clean N -complexes are precisely absolutely clean complex.

2) The class of absolutely clean N -complexes is closed under direct products, summands and direct limits.

3) $\Theta^n C$ is absolutely clean for any absolutely clean N -complex C and $n \in \mathbb{Z}$.

4) The class of absolutely clean N -complexes is coresolving. For any absolutely clean N -complex X and F of type FP_∞ , we have $\text{Ext}_{\mathcal{C}_N(R)}^n(F, X) = 0$ for all $n > 0$.

5) The class of absolutely clean N -complexes is closed under clean N -subcomplexes and clean quotients.

It is well known that an N -complex X is injective if and only if X is N -exact and $Z_n^i(X)$ is an injective module for each $n \in \mathbb{Z}$ and $i = 1, 2, \dots, N$. In the following, we will generalize this characterization to the absolutely clean N -complexes. Firstly, we need to make the following preparations.

Lemma 1 An N -complex X is absolutely clean if and only if $\text{Ext}_{\mathcal{C}_N(R)}^1(D_n^i(Q), X) = 0$ for all modules Q of type FP_∞ and $n \in \mathbb{Z}$ and $t = 1, 2, \dots, N$.

Proof Sufficiency. It is trivial.

Necessity. Without losing generality, we take P has the following form:

$$P = \dots \rightarrow 0 \rightarrow P^n \xrightarrow{d_n} P^{n-1} \xrightarrow{d_{n-1}} \dots \rightarrow P^0 \rightarrow 0 \rightarrow \dots \quad (5)$$

with each P^i a module of type FP_∞ for $0 \leq i \leq n$.

For any $0 \leq k \leq n$, we put

$$P(k) = \dots \rightarrow 0 \rightarrow P^k \xrightarrow{d_k} P^{k-1} \xrightarrow{d_{k-1}} \dots \rightarrow P^0 \rightarrow 0 \rightarrow \dots \quad (6)$$

Then $P = P(n)$. In the following, we prove the result by using induction on k . For $k=0$, by the assumption, $\text{Ext}_{\mathcal{C}_N(R)}^1(P(0), X) = 0$. Assume that $\text{Ext}_{\mathcal{C}_N(R)}^1(P(n-1), X) = 0$. Applying the functor $\text{Hom}_{\mathcal{C}_N(R)}(-, X)$ to the exact sequence

$$0 \rightarrow P(n-1) \rightarrow P(n) \rightarrow D_n^1(P^n) \rightarrow 0 \quad (7)$$

yields the following exact sequence

$$\begin{aligned} 0 &= \text{Ext}_{\mathcal{C}_N(R)}^1(D_n^1(P^n), X) \rightarrow \text{Ext}_{\mathcal{C}_N(R)}^1(P(n), X) \\ &\rightarrow \text{Ext}_{\mathcal{C}_N(R)}^1(P(n-1), X) = 0. \end{aligned} \quad (8)$$

Then $\text{Ext}_{\mathcal{C}_N(R)}^1(P(n), X)$, as desired.

Lemma 2^[11] Let $M \in R\text{-Mod}$, $X, Y \in \mathcal{C}_N(R)$ and $n \in \mathbb{Z}, i = 1, 2, \dots, N$. Then we have the following natural isomorphisms:

- 1) $\text{Hom}_{\mathcal{C}_N(R)}(D_n^N(M), Y) \cong \text{Hom}_R(M, Y_n)$.
- 2) $\text{Hom}_{\mathcal{C}_N(R)}(X, D_{n+N-1}^N(M)) \cong \text{Hom}_R(X_n, M)$.
- 3) $\text{Hom}_{\mathcal{C}_N(R)}(D_n^i(M), Y) \cong \text{Hom}_R(M, Z_n^i(Y))$.
- 4) $\text{Hom}_{\mathcal{C}_N(R)}(X, D_n^i(M)) \cong \text{Hom}_R(X_{n-(i-1)/B_{n-(i-1)}}^i(X), M)$.
- 5) $\text{Ext}_{\mathcal{C}_N(R)}^1(D_n^N(M), Y) \cong \text{Ext}_R^1(M, Y_n)$.
- 6) $\text{Ext}_{\mathcal{C}_N(R)}^1(X, D_{n+N-1}^N(M)) \cong \text{Ext}_R^1(X_n, M)$.
- 7) If Y is N -exact, then $\text{Ext}_{\mathcal{C}_N(R)}^1(D_n^i(M), Y) \cong \text{Ext}_R^1(M, Z_n^i(Y))$.
- 8) If X is N -exact, then $\text{Ext}_{\mathcal{C}_N(R)}^1(X, D_n^i(M)) \cong \text{Ext}_R^1(X_{n-(i-1)/B_{n-(i-1)}}^i(X), M)$.

Following Ref. [5], let C and D be N -complexes of left R -modules. We will denote by $\text{Hom}_R(C, D)$ the sequence of Abelian groups with $\text{Hom}_R(C, D)_n = \prod_{t \in \mathbb{Z}} \text{Hom}_R(C_t, D_{n+t})$ and such that if $f \in \text{Hom}_R(C, D)_n$ then $(d_n(f))_m = d_{n+m}^D f_m - (q^n f_{m-1})_m^C$, where q is a N -th

root of unity, $q^N = 1$ and $q \neq 1$. Then $\text{Hom}_R(C, D)$ is also an N -complex. f is called a chain map of degree n if $d_n(f) = 0$. A chain map of degree 0 is called a morphism.

Lemma 3^[13] Let C and D be N -complexes. Then $H_n^1 \text{Hom}_R(C, D) \cong \text{Hom}_{\mathcal{C}_N(R)}(C, \Theta^{-n} D)$. In particular, $\text{Hom}_R(C, D)$ is N -exact if and only if $\text{Hom}_{\mathcal{C}_N(R)}(C, D) = 0$, i. e., $H_n^t \text{Hom}_R(C, D) = 0$ is equivalent to $\text{Hom}_{\mathcal{C}_N(R)}(C, \Theta^{-n} D) = 0$ for $n \in \mathbb{Z}$ and $t = 1, 2, \dots, N$.

Proof of Theorem 1

Sufficiency. Let X be an absolutely clean N -complex. Then $\text{Ext}_{\mathcal{C}_N(R)}^1(\Sigma D_n^i(P), \Theta^{-n} X) = 0$ for any module P of type FP_∞ , $n \in \mathbb{Z}$ and $i = 1, 2, \dots, N$. This implies that N -complex $\text{Hom}_R(D_n^i(P), X)$ is N -exact^[13]. Note that P is of type FP_∞ . It follows that X must be an N -exact N -complex. Also, $\text{Ext}_{\mathcal{C}_N(R)}^1(D_n^i(P), X) = 0$ for all modules P of type FP_∞ , $n \in \mathbb{Z}$ and $i = 1, 2, \dots, N$. Using Lemma 2, we get $Z_n^i(X)$ is an absolutely clean module for $n \in \mathbb{Z}$ and $i = 1, 2, \dots, N$.

Necessity. It follows by Lemma 1 and Lemma 2.

Then the following result can be obtained by Theorem 1.

Corollary 1 If an N -complex X is absolutely clean, then $Z_n^i(X)$ is an absolutely clean module for each $n \in \mathbb{Z}$ and $i = 1, 2, \dots, N$. Moreover, X_n is an absolutely clean module for each $n \in \mathbb{Z}$.

Corollary 2 A module M is absolutely clean if and only if the N -complex $D_n^i(M)$ is absolutely clean for each $n \in \mathbb{Z}$ and $t = 1, 2, \dots, N$.

In the following, we obtain a result for bounded above N -complex X .

Proof of Proposition 1

Sufficiency. Obviously, X is N -exact and X_n is an absolutely clean module for $n \in \mathbb{Z}$ by Theorem 1 and Corollary 1.

Necessity. Assume that X has the following form:

$$\dots \rightarrow 0 \rightarrow X_0 \xrightarrow{d_0} X_{-1} \xrightarrow{d_{-1}} X_{-2} \xrightarrow{d_{-2}} X_{-3} \rightarrow \dots \quad (9)$$

Since X is N -exact,

$$\text{Ker}(d_0) = \text{Ker}(d_{-1}d_0) = \dots = \text{Ker}(d_{-N+2} \dots d_0) = 0. \quad (10)$$

This implies that $Z_0^i(X) = 0$ for $-N+1 \leq i \leq -1$ and $Z_0^{-N}(X) = X_0$. Also we have

$$\text{Ker}(d_{-1}) = \text{Ker}(d_{-2}d_{-1}) = \dots = \text{Ker}(d_{-N+2} \dots d_{-1}) = 0. \quad (11)$$

Then $Z_{-1}^i(X) = 0$ for $-N+2 \leq i \leq -1$ and $Z_{-1}^{-N+1}(X) \cong X_0$, $Z_{-1}^{-N}(X) \cong X_{-1}$.

Similarly, $Z_{-2}^i(X) = 0$ for $-N+3 \leq i \leq -1$ and $Z_{-2}^{-N+2}(X) \cong X_0$, $Z_{-2}^{-N+1}(X) \cong X_{-1}$, $Z_{-2}^{-N}(X) \cong X_{-2}$

⋮

$$\begin{aligned} Z_{-N+1}^{-1}(X) &\cong X_0, Z_{-N+1}^{-2}(X) \cong X_{-1}, \dots, \\ Z_{-N+1}^{-N+1}(X) &\cong X_{-N+2}, Z_{-N+1}^{-N}(X) \cong X_{-N+1}. \end{aligned} \quad (12)$$

Then $Z_i^t(X)$ is absolutely clean for $-N+1 \leq t \leq 0$, $i = -1, \dots, -N$.

Note that

$$\begin{aligned} 0 &\rightarrow X_0 \xrightarrow{d_0} X_{-1} \rightarrow B_{-N}^{-N+1}(X) \rightarrow 0, \\ 0 &\rightarrow X_0 \xrightarrow{d_{-1}d_0} X_{-2} \rightarrow B_{-N}^{-N+2}(X) \rightarrow 0, \\ &\vdots \\ 0 &\rightarrow X_0 \xrightarrow{d_{-N+3} \cdots d_{-1}d_0} X_{-N+2} \rightarrow B_{-N}^{-2}(X) \rightarrow 0, \\ 0 &\rightarrow X_0 \xrightarrow{d_{-N+2} \cdots d_{-1}d_0} X_{-N+1} \rightarrow B_{-N}^{-1}(X) \rightarrow 0 \end{aligned} \quad (13)$$

are exact. This implies that $Z_{-N}^i(X)$ is absolutely clean for $i = -1, \dots, -N$.

Using exact sequences

$$\begin{aligned} 0 &\rightarrow Z_{-N}^{-1}(X) \rightarrow X_{-N} \rightarrow Z_{-N-1}^{-N+1}(X) \rightarrow 0, \\ 0 &\rightarrow Z_{-N}^{-2}(X) \rightarrow X_{-N} \rightarrow Z_{-N-1}^{-N+2}(X) \rightarrow 0, \\ &\vdots \\ 0 &\rightarrow Z_{-N}^{-N+2}(X) \rightarrow X_{-N} \rightarrow Z_{-N-1}^{-2}(X) \rightarrow 0, \\ 0 &\rightarrow Z_{-N}^{-N+1}(X) \rightarrow X_{-N} \rightarrow Z_{-N-1}^{-1}(X) \rightarrow 0, \end{aligned} \quad (14)$$

we can obtain $Z_{-N-1}^i(X)$ is absolutely clean for $i = -1, \dots, -N$.

We also obtain $Z_i^t(X)$ is absolutely clean for $t \leq -N-2$, $i = -1, \dots, -N$, by a similar method. Thus X is absolutely clean by Theorem 1.

As we know, absolutely clean N -complex is important for the characterization of Gorenstein AC-injective N -complexes. An N -complex X is Gorenstein AC-injective if and only if X_n is a Gorenstein AC-injective module and $\text{Hom}_R(A, X)$ is N -exact for any absolutely clean N -complex A ^[9]. Next we obtain an equivalent characterization for N -exact absolutely clean N -complex in Gorenstein AC-injective N -complexes.

Proof of Proposition 2

Sufficiency. From the definition of the Gorenstein AC-injective N -complexes^[8], there is an exact sequence of injective N -complexes

$$I = \dots \rightarrow I^1 \xrightarrow{f^1} I^0 \xrightarrow{f^0} I^{-1} \xrightarrow{f^{-1}} \dots \quad (15)$$

with $X = \text{Ker}(f^0)$ and which remains exact after applying $\text{Hom}_{C_N(R)}(A, -)$ for any absolutely clean N -complex A . Since $\text{Ker}(f^0) = X$ and I^i are N -exact, we have $\text{Ker}(f^i)$ is N -exact for all $i \in \mathbb{Z}$.

For the exact sequence

$$0 \rightarrow \text{Ker}(f^1) \rightarrow I^1 \rightarrow X \rightarrow 0, \quad (16)$$

according to Ref. [14] we have the following exact sequence

$$0 \rightarrow Z_n^t(\text{Ker}(f^1)) \rightarrow Z_n^t(I^1) \rightarrow Z_n^t(X) \rightarrow 0. \quad (17)$$

Thus there is an exact sequence of injective modules

$$\dots \rightarrow Z_n^t(I^1) \rightarrow Z_n^t(I^0) \rightarrow Z_n^t(I^{-1}) \rightarrow \dots \quad (18)$$

with $Z_n^t(X) \cong \text{Ker}(Z_n^t(I^0) \rightarrow Z_n^t(I^{-1}))$. Next we only show that $\text{Hom}_R(M, -)$ applying the sequence (18) exact for any absolutely clean module M .

Let M be an absolutely clean module and $g: M \rightarrow Z_n^t(X)$ be a morphism of modules. Since $\text{Hom}_R(D_n^1(M), X)$ is N -exact, there exists a morphism $f: M \rightarrow X_{n+N-t}$ such that the following diagram commute (Fig. 1).

$$\begin{array}{ccccccc} & & & & M & & \\ & & & & \downarrow g & & \\ 0 & \rightarrow & Z_{n+N-t}^{N-t}(X) & \xrightarrow{\alpha} & X_{n+N-t} & \xrightarrow{p} & Z(X) \rightarrow 0 \\ & & & & \nwarrow f & & \uparrow \\ & & & & & & \end{array}$$

Fig. 1 The commutative diagram of $pf=g$

Note that I is $\text{Hom}_{C_N(R)}(D_n^t(M), -)$ exact since $D_n^t(M)$ is an absolutely clean N -complex by Corollary 2. Thus, it follows the exact sequence (16), there is an exact sequence

$$\begin{aligned} 0 &\rightarrow \text{Hom}_{C_N(R)}(D_n^t(M), \text{Ker}(f^1)) \\ &\rightarrow \text{Hom}_{C_N(R)}(D_n^t(M), I^1) \rightarrow \text{Hom}_{C_N(R)}(D_n^t(M), X) \rightarrow 0. \end{aligned} \quad (19)$$

It follows from Lemma 2

$$\begin{aligned} 0 &\rightarrow \text{Hom}_R(M, Z_n^t(\text{Ker}(f^1))) \\ &\rightarrow \text{Hom}_R(M, Z_n^t(I^1)) \rightarrow \text{Hom}_R(M, Z_n^t(X)) \rightarrow 0 \end{aligned} \quad (20)$$

is exact. Similarly, we can prove that

$$\begin{aligned} 0 &\rightarrow \text{Hom}_R(M, Z_n^t(\text{Ker}(f^i))) \\ &\rightarrow \text{Hom}_R(M, Z_n^t(I^i)) \rightarrow \text{Hom}_R(M, Z_n^t(\text{Ker}(f^{i-1}))) \rightarrow 0 \end{aligned} \quad (21)$$

is exact. Hence, $\text{Hom}_R(M, -)$ applying the sequence (18) exact.

Necessity. Note that there is an exact sequence

$$0 \rightarrow Z_{n+N-t}^{N-t}(X) \rightarrow X_n \rightarrow Z_n^t(X) \rightarrow 0 \quad (22)$$

and $Z_{n+N-t}^{N-t}(X)$ and $Z_n^t(X)$ are Gorenstein AC-injective modules, this implies X_n is a Gorenstein AC-injective module for each $n \in \mathbb{Z}$. Next we need to prove that $\text{Hom}_R(M, X)$ is N -exact for any absolutely clean N -complex M . The proof is similar to Theorem 3.2 in Ref. [2]. Thus X is a Gorenstein AC-injective N -complex.

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绝对 Clean N -复形

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摘要: 研究了绝对 clean N -复形的概念, 证明了 N -复形 X 是绝对 clean 的当且仅当 X 是 N -正合的并且对任意的 $n \in \mathbb{Z}$ 和 $i=1, 2, \dots, N$, $Z'_n(X)$ 是绝对 clean 模。特别地, 证明了上有界 N -复形 X 是绝对 clean 的当且仅当 X 是 N -正合的并且对任意的 $n \in \mathbb{Z}$, X_n 是绝对 clean 模。证明了一定条件下 N -复形 X 是 Gorenstein AC-内射的当且仅当对任意的 $n \in \mathbb{Z}$ 和 $t=1, 2, \dots, N$, $Z'_n(X)$ 是 Gorenstein AC-内射模。

关键词: 绝对 clean 模; 绝对 clean N -复形; Gorenstein AC-内射 N -复形

□