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Numerical Integration of the Function with Twin Boundary Layers

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Abstract: Traditional numerical integration requires sufficient smoothness of the integrand to achieve high-order algebraic accuracy. If the function has a boundary layer with large gradient, the composite integration formula on the uniform mesh will produce very large integration errors. In this paper, we study the Newton-Cotes formula based on Lagrange interpolation functions, local L^2 projection approximating the integrand, and Gauss integration. On the Shishkin mesh, we establish an optimal-order integration error estimate uniformly in the perturbation parameter. The convergence rate is the same as that for the smooth function. Numerical experiments confirm the sharpness of our theoretical results.

Key words: boundary layers; Shishkin mesh; Newton-Cotes formula; local L^2 projection; Gauss integration

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0 Introduction

The definite integration plays an important role in calculus. It is usually calculated by the well-known Newton-Leibniz formula. However, when the integrand cannot be represented by fundamental functions, the primitive function is complicated or the integrand has no analytic expression (e.g., some discrete point values are known), numerical integration becomes a practical necessity.

To date, the common numerical integrations are based on the composite Newton-Cotes formula. If the integrand is sufficiently smooth, the error of the composite trapezoidal and Simpson rule can attain $O(h^2)$ and $O(h^4)$, where h is the cell size. Otherwise, the composite Newton-Cotes formula based on the uniform mesh will produce very large errors.

Consider the numerical approximation for the fol-

lowing definite integration

$$I(u) = \int_0^1 u(x) dx. \quad (1)$$

Assume that the function $u(x)$ can be decomposed as

$$u(x) = \bar{u}(x) + u_{e,l}(x) + u_{e,r}(x), \quad x \in (0, 1), \quad (2)$$

where $\bar{u}(x)$ is the regular component, $u_{e,l}(x)$ and $u_{e,r}(x)$ are the two boundary layer components, which satisfy

$$\begin{cases} |\bar{u}^{(j)}(x)| \leq C, \\ |u_{e,l}^{(j)}(x)| \leq C\varepsilon^{-j/2} e^{-\beta x/\sqrt{\varepsilon}}, \\ |u_{e,r}^{(j)}(x)| \leq C\varepsilon^{-j/2} e^{-\beta(1-x)/\sqrt{\varepsilon}}, \end{cases} \quad (3)$$

for any $x \in [0, 1]$, where $\varepsilon \ll 1$ is a small positive parameter, β is a given positive constant and $j \geq 0$ is an integer. It is obvious that the function $u(x)$ exhibits a typical boundary layer, namely it has large gradients at the boundary ends $x=0$ and $x=1$.

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Such function of (2) can be viewed as the solution of the singularly perturbed reaction-diffusion problem

$$-\varepsilon u''(x) + b(x)u(x) = f(x), \quad u(0) = u(1) = 0, \quad (4)$$

where $b(x) > \beta^2 > 0$, $b(x)$ and $f(x)$ are smooth functions. The problem (4) has wide applications in such fields as optimal control, chemical reactions, fluid dynamics and so on. It is of great scientific significance to study the corresponding numerical solutions^[1].

Regarding the solution to the singularly perturbed convection-diffusion problem, there are abundant results on the numerical integration. For example, it was shown in Refs. [2-3] that the composite trapezoidal and Simpson rule on the uniform mesh will lead to an integration error of order $O(h)$. Ref. [4] studied the Lagrange interpolation of piecewise polynomials of degree k and the composite Newton-Cotes formula on a piecewise-uniform Shishkin mesh^[5]. The convergence rate is $O((N^{-1} \ln N)^{k+1})$, where N is the number of total elements. In Ref. [6], the results were extended to the Bakhvalov mesh^[7-8], the interpolation error was proved to be $O(N^{-(k+1)})$ and the integration error was $O(N^{-k})$. Furthermore, a new k point interpolation formula was discussed in Ref. [9]. The interpolation error was proved to be $O(h^k)$ when $\varepsilon = 1$, but increased to $O(h^{k-1})$ if ε is small. Ref. [10] showed the error of the Gauss integral formula using k nodes on the Shishkin mesh, the convergence rate is $O(N^{-2k})$.

However, all the above results were obtained for the integrand with one-sided boundary layer. There are very few results on the numerical integration for the solution of the singularly perturbed reaction-diffusion problem (4) with twin boundary layers. In this paper, we address this problem and present three types of numerical integrations on the Shishkin mesh. We derive some error estimates that are uniform in the perturbation parameter ε . The contributions of this paper are the following:

1) We prove that the Lagrange interpolation on the Shishkin mesh converges at a rate of order $O((N^{-1} \ln N)^{k+1})$, when piecewise polynomials of degree k and total element number N are used. Consequently, we obtain an integral error $O((N^{-1} \ln N)^{k+1})$ for the Newton-Cotes formula on the Shishkin mesh, see Theorems 1 and 2.

2) Based on the local L^2 projection for the integrand, we obtain an integral error of order $O((N^{-1} \ln N)^{k+1})$, see Theorems 3 and 4.

3) We prove that the Gauss integral formula using k

nodes on the Shishkin mesh can attain a convergence order $O(N^{-2k})$, see Theorem 5.

1 Shishkin Mesh

Define the mesh transition parameter

$$\sigma = \min \left\{ \frac{1}{4}, \frac{\tau \sqrt{\varepsilon}}{\beta} \ln N \right\}, \quad (5)$$

where $\tau > 0$ is a user-chosen parameter whose value will be discussed later. The mesh points are given by

$$x_i = \begin{cases} 4\sigma \frac{i}{N}, & i = 0, 1, \dots, \frac{N}{4}, \\ \sigma + 2(1-2\sigma) \left(\frac{i}{N} - \frac{1}{4} \right), & i = \frac{N}{4} + 1, \frac{N}{4} + 2, \dots, \frac{3N}{4}, \\ 1 - 4\sigma \left(1 - \frac{i}{N} \right), & i = \frac{3N}{4} + 1, \frac{3N}{4} + 2, \dots, N. \end{cases}$$

Set the Shishkin mesh as $\Omega_N = \{I_i\}_{i=1}^N$ with element $I_i = [x_{i-1}, x_i]$. Set the mesh size $h_i = x_i - x_{i-1}$ and then we have

$$h_i = \begin{cases} h: = \frac{4\sigma}{N}, & i = 1, 2, \dots, \frac{N}{4}, \frac{3N}{4} + 1, \dots, N, \\ H: = \frac{2(1-2\sigma)}{N}, & i = \frac{N}{4} + 1, \frac{N}{4} + 2, \dots, \frac{3N}{4}. \end{cases} \quad (6)$$

Denote $\Omega_N = \Omega^c \cup \Omega^f$, where $\Omega^c := [\sigma, 1 - \sigma]$ is the coarse domain and $\Omega^f := [0, \sigma] \cup [1 - \sigma, 1]$ is the refined domain. Figure 1 shows a Shishkin mesh for $N = 16$.

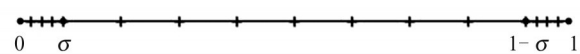


Fig. 1 The Shishkin mesh ($N=16$)

2 The Newton-Cotes Formula and Its Error Estimate

The common numerical integration is based on the Newton-Cotes formula using the Lagrange interpolation for the integrand. However, for the function (2) with boundary layer, the traditional method on the uniform mesh will result in a large error. This section presents the Lagrange interpolation and the Newton-Cotes formula on the Shishkin mesh. We will establish optimal-order error estimates that are independent of the small parameter ε .

2.1 The Lagrange Interpolation

Assume that N is a multiple of $4k$, and divide the domain $[0, 1]$ into N/k subintervals

$$[0, 1] = \bigcup_{i=0, k}^{N-k} K_i := \bigcup_{i=0, k}^{N-k} [x_i, x_{i+k}],$$

where $\bigcup_{i=0,k}^{N-k} K_i$ represents the set of cell $K_0, K_k, K_{2k}, \dots$,

K_{N-k} . On each interval $K_i=[x_i, x_{i+k}]$, we define the Lagrange interpolation of $u(x)$ by

$$L_{i,k}(u) = \sum_{n=i}^{i+k} u_n P_{n,i}(x) := \sum_{n=i}^{i+k} u_n \prod_{\substack{j=i \\ j \neq n}}^{i+k} \frac{x-x_j}{x_n-x_j}, \quad (7)$$

where $P_{n,i}(x)$ is the base function with degree k .

Define the Lebesgue constant of K_i as

$$\lambda_{i,k+1} = \max_{x \in K_i} \sum_{n=i}^{i+k} |P_{n,i}(x)|. \quad (8)$$

Since N is a multiple of $4k$, we have $K_i \subseteq \Omega^c$ or $K_i \subseteq \Omega^f$. Therefore, the subinterval $K_i=[x_i, x_{i+k}]$ is uniform. From Ref. [11], one has

$$2^{k-2}(k+1)^{-3/2} \leq \lambda_{i,k+1} \leq 2^k. \quad (9)$$

Lemma 1 Let h_i be the mesh size of the interval K_i . Denote $\omega_{k+1}(x) = (x-x_i)(x-x_{i+1}) \cdots (x-x_{i+k})$, then one has

$$|\omega_{k+1}(x)| \leq \frac{1}{4} h_i^{k+1} k!, \quad \forall x \in K_i. \quad (10)$$

Proof Without loss of generality, let $x \in K_i = [x_i, x_{i+k}]$. For any $j=i+2, i+3, \dots, i+k$, one has $|x-x_j| \leq (j-i)h_i$ and

$$\begin{aligned} |(x-x_i)(x-x_{i+1})| &= (x-x_i)(x_{i+1}-x) \\ &\leq \left(\frac{x-x_i+x_{i+1}-x}{2} \right)^2 = \frac{h_i^2}{4}, \end{aligned}$$

thus $|\omega_{k+1}(x)| \leq \frac{h_i^2}{4} \times 2h_i \times 3h_i \times \cdots \times kh_i = \frac{1}{4} h_i^{k+1} k!$.

This finishes the proof.

2.2 Error Estimate of the Lagrange Interpolation

Lemma 2 One has the following interpolation error

$$|u(x) - L_{i,k}(u)| \leq \frac{h_i^{k+1}}{4(k+1)} \max_{x \in [x_i, x_{i+k}]} |u^{(k+1)}(x)|. \quad (11)$$

Proof From Ref. [12], one has

$$|u(x) - L_{i,k}(u)| \leq \frac{|\omega_{k+1}(x)|}{(k+1)!} \max_{x \in [x_i, x_{i+k}]} |u^{(k+1)}(x)|. \quad (12)$$

Then (11) follows from Lemma 1.

Theorem 1 Suppose that $u(x)$ satisfies (2) and (3) and set $\tau=k+1$. Then one has the following interpolation error estimates.

If $\sigma < 1/4$ and $K_i = [x_i, x_{i+k}] \subseteq \Omega^f$, then

$$|u(x) - L_{i,k}(u)| \leq C(N^{-1} \ln N)^{k+1}. \quad (13)$$

If $\sigma < 1/4$ and $K_i = [x_i, x_{i+k}] \subseteq \Omega^c$, then

$$|u(x) - L_{i,k}(u)| \leq CN^{-(k+1)}. \quad (14)$$

If $\sigma = 1/4$, then

$$\begin{aligned} &|u(x) - L_{i,k}(u)| \\ &\leq C \min \left\{ \left(\sqrt{\varepsilon} N \right)^{-(k+1)}, (N^{-1} \ln N)^{k+1} \right\}. \end{aligned} \quad (15)$$

Here the bounding constant $C > 0$ is independent of ε and N .

Proof From (3) and (11), one has

$$|\bar{u}(x) - L_{i,k}(\bar{u})| \leq Ch_i^{k+1}, \quad x \in K_i,$$

then

$$|\bar{u}(x) - L_{i,k}(\bar{u})| \leq CN^{-(k+1)}, \quad x \in K_i. \quad (16)$$

To bound the interpolation error for the left boundary layer component $u_{e,l}(x)$, we proceed from the following two situations.

Case 1 $\sigma < 1/4$

For $K_i \subseteq \Omega^f$, it follows from (3) and (6) that

$$h_i = h = \frac{4\sigma}{N} = \frac{4(k+1)\sqrt{\varepsilon} \ln N}{\beta N}. \quad (17)$$

Noticing

$$|u_{e,l}^{(k+1)}(x)| \leq C\varepsilon^{-(k+1)/2} e^{-\beta x/\sqrt{\varepsilon}}, \quad x \in [x_i, x_{i+k}] \quad (18)$$

and Lemma 2, one obtains

$$|u_{e,l}(x) - L_{i,k}(u_{e,l})| \leq C \left(h/\sqrt{\varepsilon} \right)^{k+1} e^{-\beta x/\sqrt{\varepsilon}}, \quad (19)$$

namely,

$$|u_{e,l}(x) - L_{i,k}(u_{e,l})| \leq C(N^{-1} \ln N)^{k+1}. \quad (20)$$

For $K_i \subseteq \Omega^c$, due to (3) and $\tau=k+1$, one obtains

$$|u_{e,l}(x)| \leq C e^{-\beta \sigma/\sqrt{\varepsilon}} = CN^{-(k+1)}. \quad (21)$$

From (7), (8) and (21), one gets

$$|L_{i,k}(u_{e,l})| \leq CN^{-(k+1)} \lambda_{i,k+1}.$$

Hence

$$\begin{aligned} |u_{e,l}(x) - L_{i,k}(u_{e,l})| &\leq |u_{e,l}(x)| + |L_{i,k}(u_{e,l})| \\ &\leq CN^{-(k+1)}(1 + \lambda_{i,k+1}). \end{aligned}$$

Using (9) yields

$$|u_{e,l}(x) - L_{i,k}(u_{e,l})| \leq CN^{-(k+1)}. \quad (22)$$

Case 2 $\sigma = 1/4$

The partition Ω_N is uniform and $h_i = N^{-1}$. By (11), one has

$$|u_{e,l}(x) - L_{i,k}(u_{e,l})| \leq CN^{-(k+1)} \varepsilon^{-(k+1)/2} = C \left(\sqrt{\varepsilon} N \right)^{-(k+1)}.$$

If $\sqrt{\varepsilon} \geq \frac{\beta}{4(k+1) \ln N}$, then

$$|u_{e,l}(x) - L_{i,k}(u_{e,l})| \leq C \min \left\{ \left(\sqrt{\varepsilon} N \right)^{-(k+1)}, (N^{-1} \ln N)^{k+1} \right\}.$$

In a similar manner, one can bound the interpolation error for the right boundary layer component $u_{e,r}(x)$. This finishes the proof.

2.3 Error Estimate of the Newton-Cotes Formula

Let

$$I(u) = \int_0^1 u(x) dx = \sum_{i=0, k}^{N-k} \int_{x_i}^{x_{i+k}} u(x) dx = \sum_{i=0, k}^{N-k} I_{i,k}(u), \quad (23)$$

where $u(x)$ satisfies (2) and (3). On the Shishkin mesh Ω_N with $\tau=k+1$, we introduce the Newton-Cotes formula with $k+1$ nodes

$$S_k(u) = \sum_{i=0, k}^{N-k} S_{i,k}(u) = \sum_{i=0, k}^{N-k} \int_{x_i}^{x_{i+k}} L_{i,k}(u) dx. \quad (24)$$

Theorem 2 Suppose that $u(x)$ satisfies (2) and (3) and set $\tau=k+1$. Then one has the following integral error estimates.

If $\sqrt{\varepsilon} < \frac{\beta}{4(k+1)\ln N}$, then

$$|I(u) - S_k(u)| \leq C \left[N^{-(k+1)} + \sqrt{\varepsilon} (N^{-1} \ln N)^{k+1} \right]. \quad (25)$$

If $\sqrt{\varepsilon} \geq \frac{\beta}{4(k+1)\ln N}$, then

$$|I(u) - S_k(u)| \leq C \min \left\{ \left(\sqrt{\varepsilon} N \right)^{-(k+1)}, (N^{-1} \ln N)^{k+1} \right\}. \quad (26)$$

Proof We proceed from the following two situations.

Case 1 $\sqrt{\varepsilon} < \frac{\beta}{4(k+1)\ln N}$

Now one has $\sigma = \min \left\{ \frac{1}{4}, \frac{\tau \sqrt{\varepsilon}}{\beta} \ln N \right\} = \frac{\tau \sqrt{\varepsilon}}{\beta} \ln N <$

$$\frac{1}{4}.$$

By (16), one has

$$\sum_{i=0, k}^{N-k} \int_{x_i}^{x_{i+k}} |\bar{u}(x) - L_{i,k}(\bar{u})| dx \leq \sum_{i=0, k}^{N-k} C h_i N^{-(k+1)} \leq C N^{-(k+1)}.$$

Using (19) and (22) yields

$$\begin{aligned} & \sum_{i=0, k}^{N-k} \int_{x_i}^{x_{i+k}} |u_{e,l}(x) - L_{i,k}(u_{e,l})| dx \\ & \leq \sum_{i=0, k}^{N/4-k} C (N^{-1} \ln N)^{k+1} \int_{x_i}^{x_{i+k}} e^{-\beta x / \sqrt{\varepsilon}} dx + \sum_{i=N/4, k}^{3N/4-k} \int_{x_i}^{x_{i+k}} C N^{-(k+1)} dx \\ & + \sum_{i=3N/4, k}^{N-k} C (N^{-1} \ln N)^{k+1} \int_{x_i}^{x_{i+k}} e^{-\beta x / \sqrt{\varepsilon}} dx \\ & \leq C \sqrt{\varepsilon} (N^{-1} \ln N)^{k+1} + C N^{-(k+1)}. \end{aligned}$$

Analogously,

$$\begin{aligned} & \sum_{i=0, k}^{N-k} \int_{x_i}^{x_{i+k}} |u_{e,r}(x) - L_{i,k}(u_{e,r})| dx \\ & \leq C \sqrt{\varepsilon} (N^{-1} \ln N)^{k+1} + C N^{-(k+1)}, \end{aligned}$$

Collecting the above estimates gives

$$|I(u) - S_k(u)| \leq C \left[N^{-(k+1)} + \sqrt{\varepsilon} (N^{-1} \ln N)^{k+1} \right]. \quad (27)$$

Case 2 $\sqrt{\varepsilon} \geq \frac{\beta}{4(k+1)\ln N}$

Now one has $\sigma = \min \left\{ \frac{1}{4}, \frac{\tau \sqrt{\varepsilon}}{\beta} \ln N \right\} = \frac{1}{4}$.

Using (15), one obtains

$$\begin{aligned} |I(u) - S_k(u)| & \leq \sum_{i=0, k}^{N-k} |I_{i,k}(u) - S_{i,k}(u)| \\ & \leq \sum_{i=0, k}^{N-k} k h_i \max_{x \in [x_i, x_{i+k}]} |u(x) - L_{i,k}(u)| \\ & \leq C N^{-(k+1)} \min \left\{ \varepsilon^{-(k+1)/2}, \ln^{k+1} N \right\} \quad (28) \end{aligned}$$

This finishes the proof.

3 Numerical Integration Based on Local L^2 Projection

It is common to use the projection polynomials to approximate the integrand. Local L^2 projection can realize element orthogonality and is widely used in the approximation theory and finite element error analyses. In this section, we discuss the numerical integration using the local L^2 projection for the integrand.

3.1 Local L^2 Projection

Let $P^k(I_i)$ be the space composed with piecewise degree of at most k on the subinterval $I_i = [x_{i-1}, x_i]$. For any $z \in L^2(I_i)$, define $\pi z \in P^k(I_i)$ by

$$\int_{I_i} \pi z \cdot v dx = \int_{I_i} z \cdot v dx, \quad \forall v \in P^k(I_i).$$

Denote $\|\cdot\|_{L^2(I_i)}$ by the usual maximum-norm on the cell I_i . We have the following projection error estimates on the Shishkin mesh.

Lemma 3^[13] There exists a constant $C > 0$ independent of u and h_i such that

$$\|u - \pi u\|_{L^2(I_i)} \leq C h_i^{k+1} \|u^{(k+1)}\|_{L^2(I_i)}, \quad (29)$$

$$\|u - \pi u\|_{L^2(I_i)} \leq C \|u\|_{L^2(I_i)}. \quad (30)$$

3.2 Error Estimate

Theorem 3 Suppose that $u(x)$ satisfies (2) and (3) and set $\tau=k+1$. Then one has the following projection error estimates.

If $\sigma < 1/4$ and $I_i = [x_{i-1}, x_i] \subseteq \Omega^f$, then

$$\|u - \pi u\|_{L^2(I_i)} \leq C (N^{-1} \ln N)^{k+1}. \quad (31)$$

If $\sigma < 1/4$ and $I_i = [x_{i-1}, x_i] \subseteq \Omega^c$, then

$$\|u - \pi u\|_{L^2(I_i)} \leq C N^{-(k+1)}. \quad (32)$$

If $\sigma = 1/4$, then

$$\|u - \pi u\|_{L^\sigma(I_i)} \leq C \min \left\{ \left(\sqrt{\varepsilon} N \right)^{-(k+1)}, (N^{-1} \ln N)^{k+1} \right\}. \quad (33)$$

Here the bounding constant $C > 0$ is independent of ε and N .

Proof Recall $u(x) = \bar{u}(x) + u_{e,l}(x) + u_{e,r}(x)$. From (3) and (29), one has

$$\|\bar{u} - \pi \bar{u}\|_{L^\sigma(I_i)} \leq C h_i^{k+1} \|\bar{u}^{(k+1)}\|_{L^\sigma(I_i)} \leq C N^{-(k+1)}. \quad (34)$$

Now we turn to bound the projection error for the left boundary layer component $u_{e,l}(x)$.

Case 1 $\sigma < 1/4$

For $I_i = [x_{i-1}, x_i] \subseteq \Omega^f$, it follows from (17) and (29) that

$$\begin{aligned} \|u_{e,l} - \pi u_{e,l}\|_{L^\sigma(I_i)} &\leq C h_i^{k+1} \|u_{e,l}^{(k+1)}\|_{L^\sigma(I_i)} \\ &\leq C (N^{-1} \ln N)^{k+1} e^{-\beta x / \sqrt{\varepsilon}}, \end{aligned} \quad (35)$$

namely,

$$\|u_{e,l} - \pi u_{e,l}\|_{L^\sigma(I_i)} \leq C (N^{-1} \ln N)^{k+1}. \quad (36)$$

For $I_i = [x_{i-1}, x_i] \subseteq \Omega^c$, the inequalities (21), (30) and $\tau = k+1$ give

$$\|u_{e,l} - \pi u_{e,l}\|_{L^\sigma(I_i)} \leq C \|u_{e,l}\|_{L^\sigma(I_i)} \leq C e^{-\beta \sigma / \sqrt{\varepsilon}} = C N^{-(k+1)}. \quad (37)$$

Case 2 $\sigma = 1/4$

The partition Ω_N is uniform and $h_i = N^{-1}$. By (11), one has

$$\|u_{e,l} - \pi u_{e,l}\|_{L^\sigma(I_i)} \leq C h_i^{k+1} \|u_{e,l}^{(k+1)}\|_{L^\sigma(I_i)} \leq C \left(\sqrt{\varepsilon} N \right)^{-(k+1)}.$$

Note that $\sigma = 1/4$ implies $\sqrt{\varepsilon} \geq \frac{\beta}{4(k+1) \ln N}$, then one gets

$$\|u_{e,l} - \pi u_{e,l}\|_{L^\sigma(I_i)} \leq C N^{-(k+1)} \min \left\{ \varepsilon^{-(k+1)/2}, (\ln N)^{k+1} \right\}. \quad (38)$$

In a similar manner, one can bound the projection error for the right boundary layer component $u_{e,r}(x)$. This finishes the proof.

Theorem 4 Define $T(u) = \sum_{i=1}^N \int_{I_i} \pi u(x) dx$. Suppose that $u(x)$ satisfies (2) and (3) and set $\tau = k+1$. Then one has the following integral error estimates.

If $\sqrt{\varepsilon} < \frac{\beta}{4(k+1) \ln N}$, then

$$|I(u) - T(u)| \leq C \left[N^{-(k+1)} + \sqrt{\varepsilon} (N^{-1} \ln N)^{k+1} \right] \quad (39)$$

If $\sqrt{\varepsilon} \geq \frac{\beta}{4(k+1) \ln N}$, then

$$|I(u) - T(u)| \leq C \min \left\{ \left(\sqrt{\varepsilon} N \right)^{-(k+1)}, (N^{-1} \ln N)^{k+1} \right\} \quad (40)$$

Proof We proceed from the following two situa-

tions.

$$\textbf{Case 1} \quad \sqrt{\varepsilon} < \frac{\beta}{4(k+1) \ln N}$$

Now one has

$$\sigma = \min \left\{ \frac{1}{4}, \frac{\tau \sqrt{\varepsilon}}{\beta} \ln N \right\} = \frac{(k+1) \sqrt{\varepsilon}}{\beta} \ln N < \frac{1}{4}.$$

From (34), one obtains

$$\sum_{i=1}^N \int_{x_{i-1}}^{x_i} |\bar{u}(x) - L_{i,k}(\bar{u})| dx \leq \sum_{i=1}^N C N^{-(k+1)} h_i \leq C N^{-(k+1)}.$$

From (35) and (37), one has

$$\begin{aligned} &\sum_{i=1}^N \int_{x_{i-1}}^{x_i} |u_{e,l}(x) - L_{i,k}(u_{e,l})| dx \\ &\leq \sum_{i=1}^{N/4} C (N^{-1} \ln N)^{k+1} \int_{x_{i-1}}^{x_i} e^{-\beta x / \sqrt{\varepsilon}} dx + \sum_{i=N/4+1}^{3N/4} \int_{x_{i-1}}^{x_i} C N^{-(k+1)} dx \\ &\quad + \sum_{i=3N/4+1}^N C (N^{-1} \ln N)^{k+1} \int_{x_{i-1}}^{x_i} e^{-\beta x / \sqrt{\varepsilon}} dx \\ &\leq C \sqrt{\varepsilon} (N^{-1} \ln N)^{k+1} + C N^{-(k+1)}. \end{aligned}$$

Likewise, one gets

$$\begin{aligned} &\sum_{i=1}^N \int_{x_{i-1}}^{x_i} |u_{e,r}(x) - L_{i,k}(u_{e,r})| dx \\ &\leq C \sqrt{\varepsilon} (N^{-1} \ln N)^{k+1} + C N^{-(k+1)}. \end{aligned}$$

Collecting the above estimates gives

$$|I(u) - T(u)| \leq C \left[N^{-(k+1)} + \sqrt{\varepsilon} (N^{-1} \ln N)^{k+1} \right]. \quad (41)$$

$$\textbf{Case 2} \quad \sqrt{\varepsilon} \geq \frac{\beta}{4(k+1) \ln N}$$

$$\text{Now one has } \sigma = \min \left\{ \frac{1}{4}, \frac{\tau \sqrt{\varepsilon}}{\beta} \ln N \right\} = \frac{1}{4}. \text{ It fol-}$$

lows from (33) that

$$\begin{aligned} |I(u) - T(u)| &\leq \sum_{i=1}^N k h_i \max_{x \in [x_{i-1}, x_i]} |u(x) - \pi u(x)| \\ &\leq C N^{-(k+1)} \min \left\{ \varepsilon^{-(k+1)/2}, \ln^{k+1} N \right\}. \end{aligned} \quad (42)$$

This finishes the proof.

4 Gauss Integration and Its Error Estimate

Gauss integration has the highest algebraic accuracy with the same number of interpolation nodes, and is favored by numerical analysts in the practical computation. In this section, we consider the Gauss integration formula on the Shishkin mesh and derive some uniform error estimates.

4.1 Gauss Integration

Define the Gauss integration formula with m nodes for the element $I_i = [x_{i-1}, x_i]$,

$$G_{i,m}(u) = \frac{x_i - x_{i-1}}{2} \sum_{n=1}^m D_n u(x_{i,n}),$$

$$x_{i,n} = \frac{x_{i-1} + x_i}{2} + \frac{x_i - x_{i-1}}{2} d_n, \quad (43)$$

where D_n is the Gauss weight and d_n is the root of the Legendre polynomial of degree m in the interval $[-1, 1]$. Consider the Gauss formula

$$G_m(u) = \sum_{i=1}^N G_{i,m}(u), \quad (44)$$

which approximates

$$I(u) = \sum_{i=1}^N \int_{I_i} u(x) dx = \sum_{i=1}^N \int_{x_{i-1}}^{x_i} u(x) dx. \quad (45)$$

In the following we assume that $\beta = 2m$ in the inequality (3) and $\tau = 2m$ in (5).

4.2 Error Estimate of the Gauss Integration

Lemma 4^[14] Let h_i be the mesh size of element $I_i = [x_{i-1}, x_i]$, then one has the following error estimate

$$\begin{aligned} & |I_i(u) - G_{i,m}(u)| \\ & \leq \frac{h_i^{2m+1} (m!)^4}{[(2m)!]^3 (2m+1)} \max_{x \in [x_{i-1}, x_i]} |u^{(2m)}(x)|. \end{aligned} \quad (46)$$

Theorem 5 Suppose that $u(x)$ satisfies (2) and (3). Then one has the following integral error estimates.

If $\sqrt{\varepsilon} < \frac{\beta}{8m \ln N}$, then

$$|I(u) - G_m(u)| \leq C \left[N^{-2m} + \sqrt{\varepsilon} (N^{-1} \ln N)^{2m} \right]. \quad (47)$$

If $\sqrt{\varepsilon} \geq \frac{\beta}{8m \ln N}$, then

$$|I(u) - G_m(u)| \leq C \min \left\{ \left(\sqrt{\varepsilon} N \right)^{-2m}, (N^{-1} \ln N)^{2m} \right\}. \quad (48)$$

Proof Due to $h_i < 2N^{-1}$ and Lemma 4, one has

$$|I_i(\bar{u}) - G_{i,m}(\bar{u})| \leq CN^{-(2m+1)},$$

namely,

$$|I(\bar{u}) - G_m(\bar{u})| \leq CN^{-2m}. \quad (49)$$

Now we deal with the integration error of $u_{e,l}(x)$.

Case 1 $\sqrt{\varepsilon} < \frac{\beta}{8m \ln N}$

Note that $\beta = \tau = 2m$, one has

$$\sigma = \min \left\{ \frac{1}{4}, \frac{\tau \sqrt{\varepsilon}}{\beta} \ln N \right\} = \min \left\{ \frac{1}{4}, \sqrt{\varepsilon} \ln N \right\} = \sqrt{\varepsilon} \ln N < \frac{1}{4}.$$

For $I_i = [x_{i-1}, x_i] \subseteq \Omega^f$, due to (3), one gets

$$h_i = \frac{8m \sqrt{\varepsilon} \ln N}{\beta}, \quad |u_{e,l}^{(2m)}(x)| \leq C \varepsilon^{-m} e^{-\beta x_i / \sqrt{\varepsilon}}. \quad (50)$$

By Lemma 4, one has

$$|I_i(u_{e,l}) - G_{i,m}(u_{e,l})| \leq C \sqrt{\varepsilon} (N^{-1} \ln N)^{2m+1} e^{-\beta x_i / \sqrt{\varepsilon}}, \quad (51)$$

hence

$$\sum_{i=1}^{N/4} |I_i(u_{e,l}) - G_{i,m}(u_{e,l})| \leq C \sqrt{\varepsilon} (N^{-1} \ln N)^{2m+1} \sum_{i=1}^{N/4} e^{-\beta x_{i-1} / \sqrt{\varepsilon}}.$$

The geometric series gives

$$\sum_{i=1}^{N/4} e^{-\beta x_{i-1} / \sqrt{\varepsilon}} \leq \frac{1}{1 - e^{-\beta h / \sqrt{\varepsilon}}} \leq C (N^{-1} \ln N)^{-1},$$

due to $1 - e^{-\beta h / \sqrt{\varepsilon}} = O(h / \sqrt{\varepsilon}) = O(N^{-1} \ln N)$.

Then, one has

$$\sum_{i=1}^{N/4} |I_i(u_{e,l}) - G_{i,m}(u_{e,l})| \leq C \sqrt{\varepsilon} (N^{-1} \ln N)^{2m}.$$

For $I_i = [x_{i-1}, x_i] \subseteq \Omega^c$, the inequality (3) gives

$$|u_{e,l}(x)| \leq C e^{-\beta \sigma / \sqrt{\varepsilon}} \leq CN^{-2m}. \quad (52)$$

Since $D_n > 0$ and $\sum_{n=1}^m D_n = 2$ due to (43), so one has

$$\begin{aligned} G_{i,m}(u_{e,l}) &= \frac{x_i - x_{i-1}}{2} \sum_{n=1}^m D_n u_{e,l}(x_{i,n}) \\ &\leq Ch_i u_{e,l}(x_{i,n}) \leq Ch_i N^{-2m}. \end{aligned} \quad (53)$$

Combining (52) and (53) yields

$$\begin{aligned} \sum_{i=N/4+1}^{3N/4} |I_i(u_{e,l}) - G_{i,m}(u_{e,l})| &\leq \sum_{i=N/4+1}^{3N/4} [|I_i(u_{e,l})| + |G_{i,m}(u_{e,l})|] \\ &\leq \sum_{i=N/4+1}^{3N/4} Ch_i N^{-2m} \leq CN^{-2m}. \end{aligned}$$

Case 2 $\sqrt{\varepsilon} \geq \frac{\beta}{8m \ln N}$

Now one has $\sigma = 1/4$. By (3) and (46), one has

$$|I_i(u) - G_{i,m}(u)| \leq C (N^{-(2m+1)} \varepsilon^m)^{-1}.$$

Since $\sqrt{\varepsilon} \geq \frac{\beta}{8m \ln N}$, one obtains

$$|I_i(u) - G_{i,m}(u)| \leq CN^{-(2m+1)} \ln^{2m} N. \quad (54)$$

Similarly, one can obtain the integration error for $u_{e,r}(x)$. This proves Theorem 5.

5 Numerical Experiments

Consider a function

$$u(x) = \cos(2\pi x) - \frac{e^{-x/\sqrt{\varepsilon}} + e^{(x-1)/\sqrt{\varepsilon}}}{1 + e^{-1/\sqrt{\varepsilon}}}, \quad x \in [0, 1], \quad (55)$$

which exhibits twin boundary layers at $x=0$ and $x=1$, see Fig. 2.

In this section, we present the numerical results for the composite Newton-Cotes formula, piecewise discontinuous L^2 projection and Gauss integration formula on the Shishkin mesh. Set $\tau = k + 1$, $\beta = 1$ for the former two cases and $\tau = \beta = 2m$ for the last case.

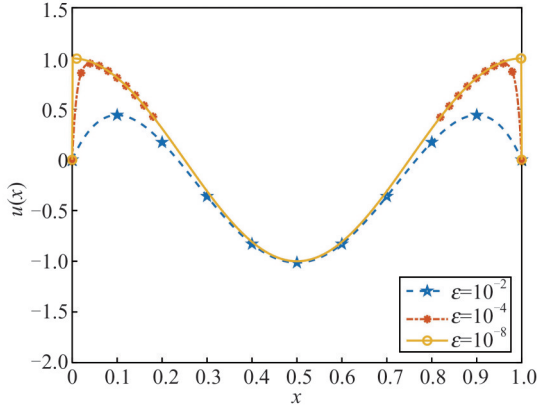


Fig. 2 A function with twin boundary layers

5.1 Newton-Cotes Formula

Consider the composite Newton-Cotes formula with four points

$$\begin{aligned} S_3(u) &= \int_{x_0}^{x_3} u(x)dx + \int_{x_3}^{x_6} u(x)dx + \cdots + \int_{x_{N-3}}^{x_N} u(x)dx \\ &= \frac{x_3 - x_0}{8} (u_0 + 3u_1 + 3u_2 + u_3) \\ &\quad + \cdots + \frac{x_N - x_{N-3}}{8} (u_{N-3} + 3u_{N-2} + 3u_{N-1} + u_N) \\ &= \sum_{i=0,3}^{N-3} \frac{3h_i}{8} (u_i + 3u_{i+1} + 3u_{i+2} + u_{i+3}), \end{aligned}$$

where $h_i = h_{i+1} = h_{i+2}$ is the cell size of the interval $[x_i, x_{i+3}]$ and the notation $\sum_{i=0,3}^{N-3}$ refers to the summation calculated from $i=0, 3, 6, \dots, N-3$. Given ε and N , we define the integration error as $\Delta_{N,\varepsilon} = |I(u) - S_3(u)|$. The convergence order is computed by

$$r = \begin{cases} \frac{\ln(\Delta_{N,\varepsilon}/\Delta_{2N,\varepsilon})}{\ln 2} & \text{on uniform grid,} \\ \frac{\ln(\Delta_{N,\varepsilon}/\Delta_{2N,\varepsilon})}{\ln(2\ln N/\ln 2N)} & \text{on Shishkin grid,} \end{cases}$$

because it shall behave as $O(N^{-r})$ and $O((N^{-1} \ln N)^r)$ for uniform and Shishkin meshes, respectively.

Table 1 shows the error and convergence rate of the integration formula $S_3(u)$ on the uniform mesh. The convergence order is generally $O(N^{-4})$. However, when $\varepsilon = 10^{-7}, 10^{-8}$, the convergence order deteriorates significantly. On the Shishkin mesh, the convergence order is restored. Table 2 shows that the convergence rate of the integration error is always $O(\sqrt{\varepsilon} (N^{-1} \ln N)^4)$. This is fully consistent with Theorem 2.

5.2 Local L^2 Projection

Consider the local L^2 projection into the space of the piecewise cubic polynomials, namely $k=3$, the projection error is defined as

$$\Delta_{N,\varepsilon} = \max_{i=1,2,\dots,N} \frac{1}{100} \sum_{j=1}^{100} [\pi u(x_{i,j}) - u(x_{i,j})].$$

Table 3 shows the projection error on the uniform mesh. When ε is large, the convergence order attains $O(N^{-4})$, but as ε decreases to zero, the convergence order deteriorates sharply. On the Shishkin mesh, the rate of convergence is robust and remains $O((N^{-1} \ln N)^4)$, as shown in Table 4. This agrees with our prediction in Theorem 3.

5.3 Gauss Integration

Consider the composite Gauss integration formula with three nodes, namely $m=3$,

$$\begin{aligned} G_3(u) &= \sum_{i=1}^N \frac{h_i}{18} \left[5u \left(\frac{x_{i-1} + x_i}{2} - \frac{h_i}{2} \sqrt{\frac{3}{5}} \right) + 8u \left(\frac{x_{i-1} + x_i}{2} \right) \right. \\ &\quad \left. + 5u \left(\frac{x_{i-1} + x_i}{2} + \frac{h_i}{2} \sqrt{\frac{3}{5}} \right) \right], \end{aligned}$$

where $h_i = x_i - x_{i-1}$. Given ε and N , define integration error as $\Delta_{N,\varepsilon} = |I(u) - G_3(u)|$.

Table 5 shows the integration error and its convergence rate on the uniform mesh. When ε is large, the rate of convergence attains $O(N^{-6})$. However, as ε becomes smaller, the convergence rate deteriorates quickly. On the Shishkin mesh, the convergence rate is improved, and is always $O(\sqrt{\varepsilon} (N^{-1} \ln N)^6)$, see Table 6. This is fully consistent with Theorem 5.

6 Conclusion

We discussed the numerical integration of the function with large gradients. The traditional numerical integration on the uniform mesh produces very large errors. We present three numerical integrations on the Shishkin mesh, i. e., the Newton-Cotes formula, local L^2 projection and Gauss integration. We derive an optimal-order error estimate independent of the small perturbation parameter. Numerical experiments confirm the sharpness of our theoretical findings. In future work, we would like to extend the results to the other types of layer-adapted meshes, such as Bakhvalov-type meshes and graded meshes^[15].

Table 1 Integration error and convergence rate using Newton-Cotes formula on the uniform mesh ($k=3$)

ε	Error/Convergence rate								
	$N=24$	$N=48$	$N=96$	$N=192$	$N=384$	$N=768$	$N=1\ 536$	$N=3\ 072$	$N=6\ 144$
1	3.48E-8/—	2.18E-9/ 4.00	1.36E-10/ 4.00	8.50E-12/ 4.00	5.31E-13/ 4.00	3.32E-14/ 4.00	2.08E-15/ 4.00	1.30E-16/ 4.00	8.11E-18/ 4.00
10^{-1}	2.18E-6/—	1.37E-7/ 3.99	8.55E-9/ 4.00	5.34E-10/ 4.00	3.34E-11/ 4.00	2.09E-12/ 4.00	1.30E-13/ 4.01	8.16E-15/ 3.99	5.10E-16/ 4.00
10^{-2}	7.23E-5/—	4.66E-6/ 3.96	2.94E-7/ 3.99	1.84E-8/ 4.00	1.15E-9/ 4.00	7.09E-11/ 4.02	4.49E-12/ 3.98	2.81E-13/ 4.00	1.75E-14/ 4.01
10^{-3}	1.68E-3/—	1.35E-4/ 3.64	9.07E-6/ 3.90	5.78E-7/ 3.97	3.63E-8/ 3.99	2.27E-9/ 4.00	1.42E-10/ 4.00	8.88E-12/ 4.00	5.55E-13/ 4.00
10^{-4}	1.27E-2/—	2.26E-3/ 2.49	2.33E-4/ 3.28	1.73E-5/ 3.75	1.13E-6/ 3.94	7.16E-8/ 3.98	4.49E-9/ 4.00	2.81E-10/ 4.00	1.75E-11/ 4.01
10^{-5}	2.49E-2/—	9.37E-3/ 1.41	2.39E-3/ 1.97	3.49E-4/ 2.78	3.13E-5/ 3.48	2.18E-6/ 3.84	1.41E-7/ 3.95	8.85E-9/ 3.99	5.54E-10/ 4.00
10^{-6}	2.92E-2/—	1.36E-2/ 1.10	5.81E-3/ 1.23	1.97E-3/ 1.56	4.20E-4/ 2.23	5.09E-5/ 3.04	4.08E-6/ 3.64	2.74E-7/ 3.90	1.74E-8/ 3.98
10^{-7}	3.06E-2/—	1.50E-2/ 1.03	7.18E-3/ 1.06	3.27E-3/ 1.13	1.32E-3/ 1.31	3.93E-4/ 1.75	6.91E-5/ 2.51	7.07E-6/ 3.29	5.22E-7/ 3.76
10^{-8}	3.11E-2/—	1.54E-2/ 1.01	7.61E-3/ 1.02	3.71E-3/ 1.04	1.75E-3/ 1.08	7.77E-4/ 1.17	2.90E-4/ 1.42	7.35E-5/ 1.98	1.06E-5/ 2.79

Table 2 Integration error and convergence rate using Newton-Cotes formula on the Shishkin mesh ($k=3$)

ε	Error/Convergence rate								
	$N=24$	$N=48$	$N=96$	$N=192$	$N=384$	$N=768$	$N=1\ 536$	$N=3\ 072$	$N=6\ 144$
10^{-7}	8.31E-5/—	1.62E-5/ 3.30	2.37E-6/ 3.64	2.81E-7/ 3.86	2.96E-8/ 3.95	2.90E-9/ 3.98	2.70E-10/ 4.00	2.42E-11/ 4.00	2.11E-12/ 4.00
10^{-8}	2.65E-5/—	5.13E-6/ 3.31	7.49E-7/ 3.64	8.90E-8/ 3.86	9.38E-9/ 3.95	9.18E-10/ 3.99	8.55E-11/ 4.00	7.67E-12/ 4.00	6.67E-13/ 4.00
10^{-9}	8.50E-5/—	1.63E-6/ 7.97	2.37E-7/ 3.65	2.82E-8/ 3.86	2.97E-9/ 3.95	2.90E-10/ 3.99	2.70E-11/ 4.00	2.43E-12/ 3.99	2.11E-13/ 4.00
10^{-10}	2.82E-6/—	5.18E-7/ 3.42	7.51E-8/ 3.65	8.91E-9/ 3.86	9.38E-10/ 3.95	9.18E-11/ 3.99	8.55E-12/ 4.00	7.67E-13/ 4.00	6.66E-14/ 4.00
10^{-11}	1.02E-6/—	1.68E-7/ 3.64	2.39E-8/ 3.69	2.82E-9/ 3.87	2.97E-10/ 3.95	2.90E-11/ 3.99	2.70E-12/ 4.00	2.43E-13/ 3.99	2.12E-14/ 4.00
10^{-12}	4.51E-7/—	5.71E-8/ 4.17	7.67E-9/ 3.80	8.96E-10/ 3.89	9.40E-11/ 3.96	9.19E-12/ 3.99	8.55E-13/ 4.00	7.69E-14/ 4.00	6.59E-15/ 4.03

Table 3 L^2 projection error and convergence rate on the uniform mesh

ε	Error/Convergence rate								
	$N=24$	$N=48$	$N=96$	$N=192$	$N=384$	$N=768$	$N=1\ 536$	$N=3\ 072$	$N=6\ 144$
1	8.15E-7/—	5.13E-8/ 3.99	3.21E-9/ 4.00	2.01E-10/ 4.00	1.26E-11/ 4.00	7.84E-13/ 4.01	4.90E-14/ 4.00	3.12E-15/ 3.97	7.68E-16/ 2.02
10^{-1}	8.35E-7/—	5.26E-8/ 3.99	3.29E-9/ 4.00	2.06E-10/ 4.00	1.29E-11/ 4.00	8.04E-13/ 4.00	5.03E-14/ 4.00	3.17E-15/ 3.99	6.17E-16/ 2.36
10^{-2}	3.48E-6/—	2.46E-7/ 3.82	1.64E-8/ 3.91	1.05E-9/ 3.97	6.69E-11/ 3.97	4.21E-12/ 3.99	2.64E-13/ 4.00	1.66E-14/ 3.99	1.15E-15/ 3.85
10^{-3}	2.78E-4/—	2.38E-5/ 3.55	1.75E-6/ 3.77	1.18E-7/ 3.89	7.71E-9/ 3.94	4.92E-10/ 3.97	3.11E-11/ 3.98	1.95E-12/ 4.00	1.22E-13/ 4.00
10^{-4}	8.04E-3/—	1.22E-3/ 2.72	1.24E-4/ 3.30	9.95E-6/ 3.64	7.07E-7/ 3.81	4.71E-8/ 3.91	3.04E-9/ 3.95	1.93E-10/ 3.98	1.22E-11/ 3.98
10^{-5}	4.37E-2/—	2.00E-2/ 1.13	4.51E-3/ 2.15	5.84E-4/ 2.95	5.37E-5/ 3.44	4.10E-6/ 3.71	2.84E-7/ 3.85	1.87E-8/ 3.92	1.20E-9/ 3.96
10^{-6}	4.35E-2/—	5.12E-2/ -0.24	3.63E-2/ 0.50	1.30E-2/ 1.48	2.37E-3/ 2.46	2.68E-4/ 3.14	2.28E-5/ 3.56	1.67E-6/ 3.77	1.13E-7/ 3.89
10^{-7}	1.61E-2/—	3.13E-2/ -0.96	4.81E-2/ -0.62	4.87E-2/ -0.02	2.79E-2/ 0.80	7.82E-3/ 1.84	1.18E-3/ 2.73	1.19E-4/ 3.31	9.52E-6/ 3.64
10^{-8}	1.02E-2/—	1.18E-2/ -0.21	2.02E-2/ -0.78	3.78E-2/ -0.90	5.08E-2/ -0.43	4.34E-2/ 0.23	1.96E-2/ 1.15	1.37E-3/ 3.84	5.62E-4/ 1.29

Table 4 L^2 projection error and convergence rate on the Shishkin mesh

ε	Error/Convergence rate								
	$N=24$	$N=48$	$N=96$	$N=192$	$N=384$	$N=768$	$N=1\ 536$	$N=3\ 072$	$N=6\ 144$
10^{-7}	1.29E-3/—	2.60E-4/ 3.23	4.03E-5/ 3.53	5.19E-6/ 3.71	5.85E-7/ 3.83	5.99E-8/ 3.91	5.75E-9/ 3.95	5.24E-10/ 3.97	4.61E-11/ 3.98
10^{-8}	1.29E-3/—	2.60E-4/ 3.23	4.03E-5/ 3.53	5.19E-6/ 3.71	5.85E-7/ 3.83	5.99E-8/ 3.91	5.75E-9/ 3.95	5.24E-10/ 3.97	4.61E-11/ 3.98
10^{-9}	1.29E-3/—	2.60E-4/ 3.23	4.03E-5/ 3.53	5.19E-6/ 3.71	5.85E-7/ 3.83	5.99E-8/ 3.91	5.75E-9/ 3.95	5.24E-10/ 3.97	4.61E-11/ 3.98
10^{-10}	1.29E-3/—	2.60E-4/ 3.23	4.03E-5/ 3.53	5.19E-6/ 3.71	5.85E-7/ 3.83	5.99E-8/ 3.91	5.75E-9/ 3.95	5.24E-10/ 3.97	4.61E-11/ 3.98
10^{-11}	1.29E-3/—	2.60E-4/ 3.23	4.03E-5/ 3.53	5.19E-6/ 3.71	5.85E-7/ 3.83	5.99E-8/ 3.91	5.75E-9/ 3.95	5.24E-10/ 3.97	4.61E-11/ 3.98
10^{-12}	1.29E-3/—	2.60E-4/ 3.23	4.03E-5/ 3.53	5.19E-6/ 3.71	5.85E-7/ 3.83	5.99E-8/ 3.91	5.75E-9/ 3.95	5.24E-10/ 3.97	4.61E-11/ 3.98

Table 5 Integration error and convergence rate using Gauss formula on the uniform mesh

ε	Error/Convergence rate						
	$N=4$	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
1	1.12E-10/—	1.75E-12/ 6.00	2.73E-14/ 6.00	4.27E-16/ 6.00	6.55E-18/ 6.03	4.06E-19/ 4.01	8.61E-20/ 2.24
10^{-1}	6.89E-8/—	1.09E-9/ 5.98	1.72E-11/ 5.99	2.68E-13/ 6.00	4.19E-15/ 6.00	6.59E-17/ 5.99	9.74E-19/ 6.08
10^{-2}	1.98E-5/—	3.59E-7/ 5.79	5.83E-9/ 5.94	9.21E-11/ 5.98	1.44E-12/ 6.00	2.26E-14/ 5.99	3.52E-16/ 6.00
10^{-3}	1.85E-3/—	7.52E-5/ 4.62	1.65E-6/ 5.51	2.83E-8/ 5.87	4.53E-10/ 5.97	7.12E-12/ 5.99	1.11E-13/ 6.00
10^{-4}	1.17E-2/—	2.81E-3/ 2.06	2.18E-4/ 3.69	6.82E-6/ 5.00	1.33E-7/ 5.68	2.21E-9/ 5.91	3.51E-11/ 5.98
10^{-5}	6.31E-3/—	5.52E-3/ 0.19	2.58E-3/ 1.10	4.23E-4/ 2.61	2.31E-5/ 4.19	5.86E-7/ 5.30	1.06E-8/ 5.79
10^{-6}	2.00E-3/—	2.00E-3/ 0.00	1.97E-3/ 0.02	1.49E-3/ 0.40	5.02E-4/ 1.57	5.59E-5/ 3.17	2.24E-6/ 4.64
10^{-7}	6.32E-4/—	6.32E-4/ 0.00	6.32E-4/ 0.00	6.32E-4/ 0.00	5.99E-4/ 0.08	3.64E-4/ 0.72	8.58E-5/ 2.08
10^{-8}	2.00E-4/—	2.00E-4/ 0.00	2.00E-4/ 0.00	2.00E-4/ 0.00	2.00E-4/ 0.00	1.99E-4/ 0.01	1.73E-4/ 0.20

Table 6 Integration error and convergence rate using Gauss formula on the Shishkin mesh

ε	Error/Convergence rate						
	$N=4$	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
10^{-7}	1.92E-5/—	6.78E-6/ 3.62	9.76E-7/ 4.78	7.80E-8/ 5.38	4.16E-9/ 5.74	1.73E-10/ 5.90	6.14E-12/ 5.97
10^{-8}	6.04E-6/—	2.14E-6/ 3.61	3.09E-7/ 4.77	2.47E-8/ 5.38	1.32E-9/ 5.73	5.47E-11/ 5.91	1.94E-12/ 5.97
10^{-9}	1.91E-6/—	6.78E-7/ 3.60	9.76E-8/ 4.78	7.80E-9/ 5.38	4.16E-10/ 5.74	1.73E-11/ 5.90	6.14E-13/ 5.97
10^{-10}	6.03E-7/—	2.14E-7/ 3.60	3.09E-8/ 4.77	2.47E-9/ 5.38	1.32E-10/ 5.73	5.47E-12/ 5.91	1.94E-13/ 5.97
10^{-11}	1.91E-7/—	6.78E-8/ 3.60	9.76E-9/ 4.78	7.80E-10/ 5.38	4.16E-11/ 5.74	1.73E-12/ 5.90	6.13E-14/ 5.97
10^{-12}	6.03E-8/—	2.14E-8/ 3.60	3.09E-9/ 4.77	2.47E-10/ 5.38	1.32E-11/ 5.73	5.47E-13/ 5.91	1.94E-14/ 5.97

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带有双边界层函数的数值积分

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摘要: 传统的数值积分需要被积函数具有较高的光滑性以获得高阶代数精度。当函数具有大梯度变化的双边界层时, 基于均匀网格的复化积分公式将产生较大的积分误差。本文研究三类数值积分公式, 即基于拉格朗日基函数的牛顿-科特斯公式、被积函数基于局部 L^2 投影近似以及经典的高斯积分。借助于分片一致的 Shishkin 网格, 证明了与小参数无关的最优阶积分误差估计, 该收敛阶与光滑函数数值积分收敛阶相同。数值实验证实了我们的理论结果。

关键词: 边界层; Shishkin 网格; 牛顿-科特斯公式; 局部 L^2 -投影; 高斯积分

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