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# Liouville Theorem for 3D Steady $Q$ -Tensor System of Liquid Crystal in Mixed Lorentz Spaces

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**Abstract:** In this paper, we study Liouville theorem for 3D steady  $Q$ -tensor system of liquid crystal in mixed Lorentz spaces. We obtain  $\mathbf{u} = \mathbf{0}, \mathbf{Q} = \mathbf{0}$  on the conditions that  $\mathbf{u} \in L_{x_1}^{p, \infty} L_{x_2}^{q, \infty} L_{x_3}^{r, \infty}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3), \mathbf{Q} \in H^2(\mathbb{R}^3), p, q, r \in (3, \infty]$ , and  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq \frac{2}{3}$ , which extends some known results.

**Key words:** mixed Lorentz spaces;  $Q$ -tensor system of liquid crystal; Liouville theorem

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## 0 Introduction

In this paper, we study the following 3D stationary  $Q$ -tensor system of liquid crystal:

$$\begin{cases} \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} + \nabla P = -\nabla \cdot (\nabla \mathbf{Q} \odot \nabla \mathbf{Q}) - \lambda \nabla \cdot (|\mathbf{Q}| \mathbf{H}) + \nabla \cdot (\mathbf{Q} \Delta \mathbf{Q} - \Delta \mathbf{Q} \mathbf{Q}), \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u} \cdot \nabla \mathbf{Q} + \mathbf{Q} \boldsymbol{\Omega} - \boldsymbol{\Omega} \mathbf{Q} - \lambda |\mathbf{Q}| \mathbf{D} = \Gamma \mathbf{H}, \end{cases} \quad (1)$$

with

$$\mathbf{H} = \Delta \mathbf{Q} - a \mathbf{Q} + b \left( \mathbf{Q}^2 - \frac{\text{tr}(\mathbf{Q}^2)}{3} I_{3 \times 3} \right) - c \mathbf{Q} \text{tr}(\mathbf{Q}^2).$$

Here  $\mathbf{u} \in \mathbb{R}^3, P \in \mathbb{R}$  and

$$\mathbf{Q} \in S_0^3 \triangleq \{ \mathbf{A} = (a_{ij})_{3 \times 3} | a_{ij} = a_{ji}, \text{tr}(\mathbf{A}) = 0 \}$$

stand for the flow velocity, the scalar pressure and the nematic tensor order parameter, respectively. The parameters  $\mu > 0, \Gamma^{-1} > 0$  and  $\lambda \in \mathbb{R}$  represent the viscosity coefficient, the rotational viscosity and the nematic alignment, respectively. The coefficients  $a, b, c \in \mathbb{R}$  with  $c > 0$  are constants.  $(\nabla \mathbf{Q} \odot \nabla \mathbf{Q})_{i,j} = \partial_{x_i} \mathbf{Q} : \partial_{x_j} \mathbf{Q}$  is the symmetric additional stress tensor.

$\mathbf{D} \triangleq \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ , and  $\boldsymbol{\Omega} \triangleq \frac{1}{2} (\nabla \mathbf{u} - \nabla \mathbf{u}^T)$  are the symmetric and skew symmetric, respectively, where the notation T represents the transposition of a matrix.

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When  $\mathbf{Q} = \mathbf{0}$ , system (1) reduces to the 3D stationary Navier-Stokes system

$$\begin{cases} \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla P = \mathbf{0}, \\ \nabla \cdot \mathbf{u} = \mathbf{0}. \end{cases} \tag{2}$$

For system (2), a well-known result on the Liouville theorem is given by Galdi<sup>[1]</sup>, which concludes if  $\mathbf{u} \in L^{\frac{9}{2}}(\mathbb{R}^3)$ , then  $\mathbf{u} = \mathbf{0}$ . Chae-Wölf<sup>[2]</sup> gave the following logarithmic improvement

$$\int_{\mathbb{R}^3} |\mathbf{u}(\mathbf{x})|^{\frac{9}{2}} \left\{ \log \left( 2 + \frac{1}{|\mathbf{u}(\mathbf{x})|} \right) \right\}^{-1} d\mathbf{x} < \infty.$$

Kozono-Terasawa-Wakasugi<sup>[3]</sup> extended Galdi's result<sup>[1]</sup> to the Lorentz spaces  $L^{\frac{9}{2}, \infty}(\mathbb{R}^3)$ . Luo and Yin<sup>[4]</sup> showed that if the solution to system (2) satisfies

$$\mathbf{u}_i \in L^{p_i}_{x_1} L^{q_i}_{x_2} L^{r_i}_{x_3}(\mathbb{R}^3), p_i, q_i, r_i \in [1, \infty), \frac{1}{p_i} + \frac{1}{q_i} + \frac{1}{r_i} = \frac{2}{3} (i=1, 2, 3), \tag{3}$$

then  $\mathbf{u} = \mathbf{0}$ , where

$$\left\| \left\| \left\| f \right\|_{L^{p_1}_{x_1}} \right\|_{L^{q_2}_{x_2}} \right\|_{L^{r_3}_{x_3}} \triangleq \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f|^p dx_1 \right)^{\frac{q}{p}} dx_2 \right)^{\frac{r}{q}} dx_3 \right)^{\frac{1}{r}}.$$

For more Liouville theorem results of system (2), one could refer to Refs. [5-9] and references therein.

In recent years, the  $\mathbf{Q}$ -tensor system of liquid crystal (1) has received much attention. However, there are few results on its Liouville theorem. Gong *et al*<sup>[10]</sup> proved that if

$$\mathbf{u} \in L^{\frac{9}{2}}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3), \mathbf{Q} \in H^2(\mathbb{R}^3), b^2 - 24ac \leq 0,$$

then  $\mathbf{u} = \mathbf{0}, \mathbf{Q} = \mathbf{0}$ . Later, Lai and Wu<sup>[11]</sup> generalized the conditions to

$$\mathbf{u} \in L^{\frac{9}{2}, \infty}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3), \mathbf{Q} \in H^2(\mathbb{R}^3), b^2 - 24ac \leq 0. \tag{4}$$

On Liouville theorem for many other models, there are also numerous results (see for example Refs. [12-15]).

Inspired by the works mentioned above in this paper, we will prove that there is only the trivial solution for the steady  $\mathbf{Q}$ -tensor system of liquid crystal model in mixed Lorentz spaces. First of all, we recall the definition of mixed Lorentz spaces.

**Definition 1** Assume the indexes  $\mathbf{p}=(p_1, p_2, p_3)$  and  $\mathbf{q}=(q_1, q_2, q_3)$  satisfying  $1 \leq p_i < \infty, 1 \leq q_i \leq \infty$ , or  $p_i = q_i = \infty (i=1, 2, 3)$ . A mixed Lorentz space  $L^{p_1, q_1}(\mathbb{R}_{x_1}; L^{p_2, q_2}(\mathbb{R}_{x_2}; L^{p_3, q_3}(\mathbb{R}_{x_3})))$  is the set of functions for which the following norm is finite:

$$\left\| \left\| \left\| f \right\|_{L^{p_1, q_1}_{x_1}} \right\|_{L^{p_2, q_2}_{x_2}} \right\|_{L^{p_3, q_3}_{x_3}} \triangleq \begin{cases} \left( \int_0^\infty \left( \int_0^\infty \left( \int_0^\infty |t^{\frac{1}{p_1}} t^{\frac{1}{p_2}} t^{\frac{1}{p_3}} f(t_1, t_2, t_3)| \left| \frac{dt_1}{t_1} \right|^{q_1} \frac{dt_2}{t_2} \right)^{\frac{p_2}{q_2}} \frac{dt_3}{t_3} \right)^{\frac{p_1}{q_1}} dt \right)^{\frac{1}{p_3}}, & 1 \leq p_i < \infty, \\ \sup_{t>0} \left\| \left\| x_3: \sup_{\gamma>0} \gamma \left\| \left\| x_2: \sup_{\lambda>0} \lambda \left\| \left\| x_1: |f(x_1, x_2, x_3)| \lambda \right\| \right|^{\frac{1}{p_1}} \gamma \right\| \right|^{\frac{1}{p_2}} \right\| t \right\|^{\frac{1}{p_3}}, & p_i = \infty. \end{cases}$$

The main result of the paper is stated in the following theorem.

**Theorem 1** Let  $(\mathbf{u}, \mathbf{Q})$  be a smooth solution to system (1) in  $\mathbb{R}^3$ . If

$$\mathbf{u} \in L^{p, \infty}_{x_1} L^{q, \infty}_{x_2} L^{r, \infty}_{x_3}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3), \mathbf{Q} \in H^2(\mathbb{R}^3), p, q, r \in (3, \infty], \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq \frac{2}{3}, \text{ and } b^2 - 24ac \leq 0,$$

then  $\mathbf{u} = \mathbf{0}, \mathbf{Q} = \mathbf{0}$ .

**Remark 1** In the case of  $p=q=r=\frac{9}{2}$  in Theorem 1, the sufficient condition coincides with (4) obtained in previous work<sup>[11]</sup>. In addition, due to the following embedding (see for example Ref. [16])

$$L^{\frac{9}{2}}(\mathbb{R}^3) \subset L^{\frac{9}{2-q}}(\mathbb{R}^3) \left(\frac{9}{2} \leq q \leq \infty\right), \tag{5}$$

we deduce that our result extends Gong *et al*'s result in Ref. [10].

When  $\mathcal{Q} = \mathbf{0}$ , we obtain the following Liouville theorem for the Navier-Stokes equations (2), which is a supplement to the result of Ref. [4] (see (2)).

**Corollary 1** Let  $\mathbf{u} \in L^{p, \infty}_{x_1} L^{q, \infty}_{x_2} L^{r, \infty}_{x_3}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$  be a smooth solution to the Navier-Stokes equations (2) in  $\mathbb{R}^3$ . If  $p, q, r \in (3, \infty], \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq \frac{2}{3}$ , then  $\mathbf{u} = \mathbf{0}$ .

### 1 Proof of Theorem 1

In this section, we give the proof of Theorem 1. To streamline the presentation, set

$$B_1 \triangleq \{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} \in (R, 2R)\}, \quad B_2 \triangleq \{x_i \in \mathbb{R} \mid R < |x_i| < 2R, i = 1, 2, 3\},$$

$$\left\| \left\| \cdot \right\|_{L^{p, \infty}_{x_1}} \right\|_{L^{q, \infty}_{x_2}} \left\| \cdot \right\|_{L^{r, \infty}_{x_3}} \left\| \cdot \right\|_{L^{p, \infty}(R|x_1|2R)} \left\| \cdot \right\|_{L^{q, \infty}(R|x_2|2R)} \left\| \cdot \right\|_{L^{r, \infty}(R|x_3|2R)}.$$

We firstly recall the following lemma.

**Lemma 1**<sup>[17]</sup> Let  $\beta(\mathcal{Q}) = 1 - 6 \frac{[\text{tr}(\mathcal{Q}^3)]^2}{|\mathcal{Q}|^6}, \mathcal{Q} \in S_0^3$ , then  $0 \leq \beta(\mathcal{Q}) \leq 1$ .

**Proof** We consider a smooth cut-off function  $\phi \in C_c^\infty(\mathbb{R})$  such that

$$\phi(y) = \begin{cases} 1, & |y| < 1, \\ 0, & |y| \geq 2. \end{cases}$$

For each  $R > 0$ , defining

$$\phi_R(\mathbf{x}) = \phi\left(\frac{x_1}{R}\right) \phi\left(\frac{x_2}{R}\right) \phi\left(\frac{x_3}{R}\right), \quad \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

then, one has

$$\phi_R(\mathbf{x}) = \begin{cases} 1, & |\mathbf{x}| < R, \\ 0, & |\mathbf{x}| \geq 2R. \end{cases}$$

Moreover, there is a constant  $C$  independent of  $R$  such that  $|\nabla^k \phi_R| \leq \frac{C}{R^k}$  for  $k \in \{0, 1, 2, 3\}$ .

Multiplying the first equation and the second equation of (1) by  $\mathbf{u}(\mathbf{x})\phi_R(\mathbf{x})$  and  $-\mathbf{H}(\mathbf{x})\phi_R(\mathbf{x})$ , respectively, we know

$$\begin{aligned} & \mu \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 \phi_R(\mathbf{x}) dx + \Gamma \int_{\mathbb{R}^3} |\mathbf{H}|^2 \phi_R(\mathbf{x}) dx \\ &= \int_{\mathbb{R}^3} (\mathbf{u} \cdot \nabla \mathcal{Q}) : \Delta \mathcal{Q} \phi_R(\mathbf{x}) dx - \int_{\mathbb{R}^3} (\mathcal{Q} \mathcal{Q} - \mathcal{Q} \mathcal{Q}) : \Delta \mathcal{Q} \phi_R(\mathbf{x}) dx - \int_{\mathbb{R}^3} (\mathbf{u} \cdot \nabla \mathcal{Q}) : \left[ a \mathcal{Q} + b \left( \mathcal{Q}^2 - \frac{\text{tr}(\mathcal{Q}^2)}{3} I_{3 \times 3} \right) + c \mathcal{Q} |\mathcal{Q}|^2 \right] \phi_R(\mathbf{x}) dx \\ &+ \int_{\mathbb{R}^3} (\mathcal{Q} \mathcal{Q} - \mathcal{Q} \mathcal{Q}) : \left[ a \mathcal{Q} + b \left( \mathcal{Q}^2 - \frac{\text{tr}(\mathcal{Q}^2)}{3} I_{3 \times 3} \right) + c \mathcal{Q} |\mathcal{Q}|^2 \right] \phi_R(\mathbf{x}) dx - \lambda \int_{\mathbb{R}^3} |\mathcal{Q}| \mathbf{D} : \mathbf{H} \phi_R(\mathbf{x}) dx - \int_{\mathbb{R}^3} \nabla \cdot (\nabla \mathcal{Q} \odot \nabla \mathcal{Q}) \cdot \mathbf{u} \phi_R(\mathbf{x}) dx \\ &+ \lambda \int_{\mathbb{R}^3} |\mathbf{x}| \mathbf{H} : \nabla (\mathbf{u} \phi_R(\mathbf{x})) dx - \int_{\mathbb{R}^3} (\mathcal{Q} \Delta \mathcal{Q} - \Delta \mathcal{Q} \mathcal{Q}) : \nabla (\mathbf{u} \phi_R(\mathbf{x})) dx + \int_{\mathbb{R}^3} \frac{1}{2} |\mathbf{u}|^2 \mathbf{u} \cdot \nabla \phi_R(\mathbf{x}) dx + \int_{\mathbb{R}^3} P \mathbf{u} \cdot \nabla \phi_R(\mathbf{x}) + \frac{\mu}{2} |\mathbf{u}|^2 \Delta \phi_R(\mathbf{x}) dx \\ &\triangleq \sum_{i=1}^{10} I_i. \end{aligned} \tag{6}$$

Firstly, since  $\mathcal{Q}$  is symmetric and  $\mathcal{Q}$  is skew symmetric, we deduce  $I_4 = 0$ . For  $I_1 + I_6, I_2 + I_8$  and  $I_5 + I_7$ , using Hölder inequality, we obtain

$$\begin{aligned} I_1 + I_6 &= \int_{\mathbb{R}^3} \mathbf{u} \nabla \mathcal{Q} : \Delta \mathcal{Q} \phi_R(\mathbf{x}) dx - \int_{\mathbb{R}^3} \nabla \cdot (\nabla \mathcal{Q} \odot \nabla \mathcal{Q}) \cdot \mathbf{u} \phi_R(\mathbf{x}) dx \\ &= \int_{\mathbb{R}^3} (\mathbf{u} \cdot \nabla \mathcal{Q}) : \nabla \mathcal{Q} \phi_R(\mathbf{x}) dx - \int_{\mathbb{R}^3} (\mathbf{u} \cdot \nabla \mathcal{Q}) \Delta \mathcal{Q} \phi_R(\mathbf{x}) dx - \int_{\mathbb{R}^3} \mathbf{u} \cdot \nabla \frac{1}{2} |\nabla \mathcal{Q}|^2 \phi_R(\mathbf{x}) dx = \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \mathcal{Q}|^2 \mathbf{u} \cdot \nabla \phi_R(\mathbf{x}) dx \end{aligned}$$

$$\begin{aligned} &\leq \frac{C}{R} \|\nabla \mathbf{Q}\|^2_{L^{2\infty}(B_1)} \|\mathbf{u}\|_{L^{p_1} \times L^{q_2} \times L^{r_3}(B_2)} \|\nabla \phi(\frac{\cdot}{R})\|_{L^{\frac{2p}{p-2},1} \times L^{\frac{2q}{q-2},1} \times L^{\frac{2r}{r-2},1}(B_2)} \\ &\leq CR^{\frac{1}{2} - (\frac{1}{p} + \frac{1}{q} + \frac{1}{r})} \|\nabla \mathbf{Q}\|^2_{L^2(B_1)} \|\mathbf{u}\|_{L^{p_1} \times L^{q_2} \times L^{r_3}(B_2)} \|\nabla \phi\|_{L^{\frac{2p}{p-2},1} \times L^{\frac{2q}{q-2},1} \times L^{\frac{2r}{r-2},1}(B_2)}, \end{aligned} \tag{7}$$

$$\begin{aligned} I_2 + I_8 &= - \int_{\mathbb{R}^3} (\mathbf{Q}\mathbf{Q} - \mathbf{Q}\mathbf{Q}) : \Delta \mathbf{Q}\phi_R(\mathbf{x}) d\mathbf{x} - \int_{\mathbb{R}^3} \nabla \cdot (\nabla \mathbf{Q}\mathbf{Q}\nabla \mathbf{Q}) \cdot \mathbf{u}\phi_R(\mathbf{x}) d\mathbf{x} = - \int_{\mathbb{R}^3} (\mathbf{Q}\Delta \mathbf{Q} - \Delta \mathbf{Q}\mathbf{Q}) : \mathbf{u}\mathbf{Q}\nabla \phi_R(\mathbf{x}) d\mathbf{x} \\ &\leq \frac{C}{R} \|\nabla^2 \mathbf{Q}\|_{L^{2\infty}(B_1)} \|\mathbf{Q}\|_{L^\infty(B_1)} \|\mathbf{u}\|_{L^{p_1} \times L^{q_2} \times L^{r_3}(B_2)} \|\nabla \phi(\frac{\cdot}{R})\|_{L^{\frac{2p}{p-2},1} \times L^{\frac{2q}{q-2},1} \times L^{\frac{2r}{r-2},1}(B_2)} \\ &\leq CR^{\frac{1}{2} - (\frac{1}{p} + \frac{1}{q} + \frac{1}{r})} \|\nabla^2 \mathbf{Q}\|_{L^2(B_1)} \|\mathbf{u}\|_{L^{p_1} \times L^{q_2} \times L^{r_3}(B_2)} \|\nabla \phi\|_{L^{\frac{2p}{p-2},1} \times L^{\frac{2q}{q-2},1} \times L^{\frac{2r}{r-2},1}(B_2)}, \end{aligned} \tag{8}$$

$$\begin{aligned} I_5 + I_7 &= -\lambda \int_{\mathbb{R}^3} |\mathbf{Q}|\mathbf{D}:\mathbf{H}\phi_R(\mathbf{x}) d\mathbf{x} + \lambda \int_{\mathbb{R}^3} |\mathbf{Q}|\mathbf{H}:\nabla(\mathbf{u}\phi_R(\mathbf{x})) d\mathbf{x} = \int_{\mathbb{R}^3} |\mathbf{Q}|\mathbf{H}:\mathbf{u} \otimes \nabla \phi_R(\mathbf{x}) d\mathbf{x} \\ &\leq \frac{C}{R} \|\mathbf{H}\|_{L^2(B_1)} \|\mathbf{Q}\|_{L^\infty(B_1)} \|\mathbf{u}\|_{L^{p_1} \times L^{q_2} \times L^{r_3}(B_2)} \|\nabla \phi(\frac{\cdot}{R})\|_{L^{\frac{2p}{p-2},1} \times L^{\frac{2q}{q-2},1} \times L^{\frac{2r}{r-2},1}(B_2)} \\ &\leq CR^{\frac{1}{2} - (\frac{1}{p} + \frac{1}{q} + \frac{1}{r})} \|\mathbf{H}\|_{L^2(B_1)} \|\mathbf{u}\|_{L^{p_1} \times L^{q_2} \times L^{r_3}(B_2)} \|\nabla \phi\|_{L^{\frac{2p}{p-2},1} \times L^{\frac{2q}{q-2},1} \times L^{\frac{2r}{r-2},1}(B_2)}. \end{aligned} \tag{9}$$

For  $I_3$  and  $I_9$ , along the same line, we have

$$\begin{aligned} I_3 &= - \int_{\mathbb{R}^3} (\mathbf{u} \cdot \nabla \mathbf{Q}) : \left[ a\mathbf{Q} + b\left(\mathbf{Q}^2 - \frac{\text{tr}(\mathbf{Q}^2)}{3} I_{3 \times 3}\right) + c\mathbf{Q}|\mathbf{Q}|^2 \right] \phi_R(\mathbf{x}) d\mathbf{x} = - \int_{\mathbb{R}^3} (\mathbf{Q}\Delta \mathbf{Q} - \Delta \mathbf{Q}\mathbf{Q}) : \mathbf{u}\mathbf{Q}\nabla \phi_R(\mathbf{x}) d\mathbf{x} \\ &\leq \frac{C}{R} \left(1 + \|\mathbf{Q}\|_{L^\infty(B_1)} + \|\mathbf{Q}\|_{L^\infty(B_1)}^2\right) \|\mathbf{u}\|_{L^{p_1} \times L^{q_2} \times L^{r_3}(B_2)} \|\mathbf{Q}\|_{L^\infty(B_1)} \|\nabla \phi(\frac{\cdot}{R})\|_{L^{\frac{2p}{p-2},1} \times L^{\frac{2q}{q-2},1} \times L^{\frac{2r}{r-2},1}(B_2)} \\ &\leq CR^{\frac{1}{2} - (\frac{1}{p} + \frac{1}{q} + \frac{1}{r})} \|\mathbf{u}\|_{L^{p_1} \times L^{q_2} \times L^{r_3}(B_2)} \|\mathbf{Q}\|_{L^\infty(B_1)}^2 \|\nabla \phi\|_{L^{\frac{2p}{p-2},1} \times L^{\frac{2q}{q-2},1} \times L^{\frac{2r}{r-2},1}(B_2)}, \end{aligned} \tag{10}$$

$$\begin{aligned} I_9 &= \int_{\mathbb{R}^3} \frac{1}{2} |\mathbf{u}|^2 \mathbf{u} \cdot \nabla \phi_R(\mathbf{x}) d\mathbf{x} \leq \frac{C}{R} \|\mathbf{u}\|_{L^{p_1} \times L^{q_2} \times L^{r_3}(B_2)}^3 \|\nabla \phi(\frac{\cdot}{R})\|_{L^{\frac{p}{p-3},1} \times L^{\frac{q}{q-3},1} \times L^{\frac{r}{r-3},1}(B_2)} \\ &\leq CR^{2 - (\frac{3}{p} + \frac{3}{q} + \frac{3}{r})} \|\mathbf{u}\|_{L^{p_1} \times L^{q_2} \times L^{r_3}(B_2)}^3 \|\nabla \phi\|_{L^{\frac{p}{p-3},1} \times L^{\frac{q}{q-3},1} \times L^{\frac{r}{r-3},1}(B_2)}. \end{aligned} \tag{11}$$

For  $I_{10}$ , note that

$$\Delta P = -\text{divdiv}(\mathbf{u} \otimes \mathbf{u} + \nabla \mathbf{Q}\mathbf{Q}\nabla \mathbf{Q} + \lambda |\mathbf{Q}|\mathbf{H} + \mathbf{Q}\Delta \mathbf{Q} - \Delta \mathbf{Q}\mathbf{Q}).$$

Let  $P = P_1 + P_2$  such that  $\Delta P_1 = -\text{divdiv}(f_1)$  and  $\Delta P_2 = -\text{divdiv}(f_2)$ , where

$$f_1 \triangleq \mathbf{u} \otimes \mathbf{u}, f_2 \triangleq \nabla \mathbf{Q}\mathbf{Q}\nabla \mathbf{Q} + \lambda |\mathbf{Q}|\mathbf{H} + \mathbf{Q}\Delta \mathbf{Q} - \Delta \mathbf{Q}\mathbf{Q}.$$

In view of the conditions  $\mathbf{u} \in L^{p_1, s_1} L^{q_2, s_2} L^{r_3, s_3}(\mathbb{R}^3)$  and  $\mathbf{Q} \in H^2(\mathbb{R}^3)$ , we obtain that

$$f_1 \in L^{p, \infty}_{x_1} L^{q, \infty}_{x_2} L^{r, \infty}_{x_3}(\mathbb{R}^3), f_2 \in L^2(\mathbb{R}^3).$$

By Calderron-Zygmund theorem, it follows that

$$P_1 \in L^{p, \infty}_{x_1} L^{q, \infty}_{x_2} L^{r, \infty}_{x_3}(\mathbb{R}^3), P_2 \in L^2(\mathbb{R}^3).$$

Then,

$$\begin{aligned} I_{10} &= \int_{\mathbb{R}^3} P \mathbf{u} \cdot \nabla \phi_R(\mathbf{x}) + \frac{\mu}{2} |\mathbf{u}|^2 \Delta \phi_R(\mathbf{x}) dx = \int_{\mathbb{R}^3} P_1 \mathbf{u} \cdot \nabla \phi_R(\mathbf{x}) dx + \int_{\mathbb{R}^3} P_2 \mathbf{u} \cdot \nabla \phi_R(\mathbf{x}) dx + \int_{\mathbb{R}^3} \frac{\mu}{2} |\mathbf{u}|^2 \Delta \phi_R(\mathbf{x}) dx \\ &\leq \frac{C}{R} \left\| \left\| \mathbf{u} \right\|_{L^{p, \infty}_{x_1} L^{q, \infty}_{x_2} L^{r, \infty}_{x_3}(B_2)} \right\|^3 \left\| \left\| \nabla \phi\left(\frac{\cdot}{R}\right) \right\|_{L^{\frac{p}{p-3}, 1}_{x_1} L^{\frac{q}{q-3}, 1}_{x_2} L^{\frac{r}{r-3}, 1}_{x_3}(B_2)} \right\| \\ &\quad + \frac{C}{R} \left\| \left\| \mathbf{u} \right\|_{L^{p, \infty}_{x_1} L^{q, \infty}_{x_2} L^{r, \infty}_{x_3}(B_2)} \right\| \left\| \left\| \nabla \phi\left(\frac{\cdot}{R}\right) \right\|_{L^{\frac{2p}{p-2}, 1}_{x_1} L^{\frac{2q}{q-2}, 1}_{x_2} L^{\frac{2r}{r-2}, 1}_{x_3}(B_2)} \right\| \left\| P_2 \right\|_{L^2(B_1)} \\ &\quad + \frac{C}{R^2} \left\| \left\| \mathbf{u} \right\|_{L^{p, \infty}_{x_1} L^{q, \infty}_{x_2} L^{r, \infty}_{x_3}(B_2)} \right\|^2 \left\| \left\| \Delta \phi\left(\frac{\cdot}{R}\right) \right\|_{L^{\frac{p}{p-2}, 1}_{x_1} L^{\frac{q}{q-2}, 1}_{x_2} L^{\frac{r}{r-2}, 1}_{x_3}(B_2)} \right\| \\ &\leq CR^{2-\left(\frac{3}{p} + \frac{3}{q} + \frac{3}{r}\right)} \left\| \left\| \mathbf{u} \right\|_{L^{p, \infty}_{x_1} L^{q, \infty}_{x_2} L^{r, \infty}_{x_3}(B_2)} \right\|^3 \left\| \left\| \nabla \phi \right\|_{L^{\frac{p}{p-3}, 1}_{x_1} L^{\frac{q}{q-3}, 1}_{x_2} L^{\frac{r}{r-3}, 1}_{x_3}(B_2)} \right\| \\ &\quad + CR^{\frac{1}{2} - \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)} \left\| \left\| \mathbf{u} \right\|_{L^{p, \infty}_{x_1} L^{q, \infty}_{x_2} L^{r, \infty}_{x_3}(B_2)} \right\| \left\| P_2 \right\|_{L^2(B_1)} \left\| \left\| \nabla \phi\left(\frac{\cdot}{R}\right) \right\|_{L^{\frac{2p}{p-2}, 1}_{x_1} L^{\frac{2q}{q-2}, 1}_{x_2} L^{\frac{2r}{r-2}, 1}_{x_3}(B_2)} \right\| \\ &\quad + CR^{1 - \left(\frac{2}{p} + \frac{2}{q} + \frac{2}{r}\right)} \left\| \left\| \mathbf{u} \right\|_{L^{p, \infty}_{x_1} L^{q, \infty}_{x_2} L^{r, \infty}_{x_3}(B_2)} \right\|^2 \left\| \left\| \nabla \phi\left(\frac{\cdot}{R}\right) \right\|_{L^{\frac{p}{p-2}, 1}_{x_1} L^{\frac{q}{q-2}, 1}_{x_2} L^{\frac{r}{r-2}, 1}_{x_3}(B_2)} \right\|. \end{aligned} \tag{12}$$

Therefore, when  $R \rightarrow \infty$ , we can obtain  $I_i \rightarrow 0 (i = 1, \dots, 10)$ . Thanks to the Sobolev embedding, we have

$$\|\mathbf{u}\|_{L^6(\mathbb{R}^3)} \leq C \|\nabla \mathbf{u}\|_{L^2(\mathbb{R}^3)},$$

then  $\mathbf{u} = \mathbf{0}$  and  $\mathbf{H} = \mathbf{0}$ . By the definition of  $\mathbf{H}$ , we observe

$$-\Delta \mathbf{Q} = -a\mathbf{Q} + b[\mathbf{Q}^2 - \frac{\text{tr}(\mathbf{Q}^2)}{3} I_{3 \times 3}] - c\mathbf{Q} \text{tr}(\mathbf{Q}^2). \tag{13}$$

Taking the inner product of the equation (13) with  $\mathbf{Q}\phi_R(\mathbf{x})$ , and by Lemma 1, one yields

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla \mathbf{Q}|^2 \phi_R(\mathbf{x}) dx &= \int_{\mathbb{R}^3} |\mathbf{Q}|^2 \Delta \phi_R(\mathbf{x}) dx - \int_{\mathbb{R}^3} [a|\mathbf{Q}|^2 - b \text{tr}(\mathbf{Q}^3) + c|\mathbf{Q}|^4] \phi_R(\mathbf{x}) dx \\ &\leq \frac{C}{R^2} \int_{\mathbb{R}^3} |\mathbf{Q}|^2 dx + \int_{\mathbb{R}^3} \left( \frac{b^2 - 24ac}{24c} \right) |\mathbf{Q}|^2 \phi_R(\mathbf{x}) dx \leq \frac{C}{R^2} \int_{\mathbb{R}^3} |\mathbf{Q}|^2 dx \rightarrow 0 (R \rightarrow \infty), \end{aligned}$$

which combined with  $\|\mathbf{Q}\|_{L^2} < \infty$ , concludes  $\mathbf{Q} = \mathbf{0}$ . The proof of Theorem 1 is completed.

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## 三维稳态 $Q$ -tensor 液晶流系统在混合 Lorentz 空间中的 Liouville 型定理

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**摘要:** 本文在混合 Lorentz 空间中研究了三维稳态  $Q$ -tensor 液晶系统的 Liouville 定理。我们在条件  $\mathbf{u} \in L_{x_1}^{p,\infty} L_{x_2}^{q,\infty} L_{x_3}^{r,\infty}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$ ,  $\mathbf{Q} \in H^2(\mathbb{R}^3)$ ,  $p, q, r \in (3, \infty]$  和  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq \frac{2}{3}$  下得到  $\mathbf{u} = \mathbf{0}$ ,  $\mathbf{Q} = \mathbf{0}$  这一结论, 它推广了一些已有的结果。

**关键词:** 混合 Lorentz 空间;  $Q$ -tensor 液晶流系统; Liouville 定理

□