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# $\Sigma$ -Shaped Connected Component of Positive Solutions for One-Dimensional Prescribed Mean Curvature Equation in Minkowski Space

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**Abstract:** In this work, we demonstrate that the existence of an  $\Sigma$ -shaped connected component within the set of positive solutions for the

one-dimensional prescribed mean curvature equation in Minkowski space 
$$\begin{cases} -\left(\frac{u'(t)}{\sqrt{1-(u'(t))^2}}\right)' = \lambda h(t)f(u(t)), t \in (0, 1), \\ u(0) = 0, u'(1) + \sqrt{\lambda} u(1) = 0, \end{cases}$$
 with boundary conditions

having parameter in two cases  $f(0)=0$  and  $f(0)>0$  by using upper and lower solution method, where  $\lambda > 0$  is a parameter,  $f \in C^2([0, \infty), \mathbb{R})$  is monotonically increasing and  $\lim_{u \rightarrow 1^-} \frac{f(u)}{1-u} = 0$ ,  $h \in C^1([0, 1], (0, \infty))$  is a nonincreasing function and  $h(t) > 1$ .

**Key words:** boundary conditions with parameters; positive solutions; the upper and lower solution method; asymptotic property

**CLC number:** O175

## 0 Introduction

The boundary value problem in which the boundary conditions involve parameters is one of the most significant problems in the mathematical theory. In 1994, Binding was the first to propose the Sturm Liouville problems with boundary conditions dependent on eigenparameters<sup>[1]</sup>. In 1999, Hai<sup>[2]</sup> used the prior estimation

method to prove the existence of solutions for boundary value problem

$$\begin{cases} u''(t) + a(t)u'(t) + f(u(t)) = 0, t \in (0, 1), \\ u(0) = 0, u(1) = \lambda \end{cases}$$

with boundary conditions having parameters. Since then, many different problems with parameters under boundary conditions have been studied respectively by Marinets *et al*<sup>[3]</sup>, and a great deal of fruitful results have been

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achieved, which can be found in Refs. [4-7]. Fonseca *et al*<sup>[6-7]</sup> investigated the positive solution of the boundary value problem with two boundary conditions having parameters by employing the upper and lower solution method. They proved the existence and multiplicity of the positive solutions and obtained the bifurcation diagram of the above positive solutions by using the time mapping method. Particularly, Fonseca *et al*<sup>[6]</sup> discussed the existence and multiplicity results of the steady-state reaction diffusion equation

$$\begin{cases} -u''(t) = \lambda h(t)f(u(t)), t \in (0, 1), \\ -du'(0) + \mu(\lambda)u(0) = 0, \\ u'(1) + \mu(\lambda)u(1) = 0 \end{cases}$$

via the upper and lower solution method in three cases of  $f(0)=0, f(0)>0$  and  $f(0)<0$ . Further, they established a unique result for  $\lambda \approx 0$  and  $\lambda \gg 1$ . In the above equation,  $\lambda > 0$  is a parameter,  $f \in C^2([0, \infty), \mathbb{R})$  is an increasing function which is sublinear at infinity,  $h \in C^1([0, 1], (0, \infty))$  is a nonincreasing function with  $h_1 := h(1) > 0$  and there exist constants  $d_0 > 0, \alpha \in [0, 1]$  such that  $h(t) \leq \frac{d_0}{t^\alpha}$  for all  $t \in (0, 1]$ , and  $\mu \in C([0, \infty), [0, \infty))$  is an increasing function such that  $\mu(0) \geq 0$ .

The prescribed mean curvature equation addressed in this paper is regarded as a significant problem in partial differential equations and possesses extensive applications in other domains like physics and biology, such as the shape of the human cornea<sup>[8-9]</sup>, drops of capillary droopiness<sup>[10]</sup>, micro electronic mechanical systems, and corresponding models of large spatial gradients<sup>[11]</sup>. Specifically, the Dirichlet problem with the Minkowski-curvature operator

$$\begin{cases} -\operatorname{div} \left( \frac{\nabla u(t)}{\sqrt{1 - (|\nabla u(t)|)^2}} \right) = f(t, u(t)), t \in \Omega, \\ u(0) = 0, t \in \partial\Omega \end{cases}$$

has drawn the attention of numerous scholars. Especially, the existence and multiplicity of positive solutions for the Dirichlet problem with the one-dimensional Minkowski-curvature operator

$$\begin{cases} - \left( \frac{u'(t)}{\sqrt{1 - (u'(t))^2}} \right)' = \lambda f(u(t)), t \in (0, 1), \\ u(0) = u(1) = 0 \end{cases}$$

have been extensively studied by employing the variational method, the upper and lower solution method, and the time mapping method. See Refs. [12-20], where  $\lambda > 0$  is parameter,  $f \in C([0, \infty), [0, \infty))$  and  $f(u) > 0 (u > 0)$ . It is

obvious that the problems in the above references merely study the positive solutions of the Dirichlet problem with the mean curvature operator in Minkowski space in the case where the nonlinear term has a parameter. However, the prescribed mean curvature boundary value problem where both the nonlinear term and the boundary conditions have parameters must be further considered to accurately describe the corresponding physical phenomena.

Inspired by the above-mentioned papers, we explore the existence and multiplicity of positive solutions for the one-dimensional prescribed mean curvature problem in Minkowski space

$$\begin{cases} - \left( \frac{u'(t)}{\sqrt{1 - (u'(t))^2}} \right)' = \lambda h(t)f(u(t)), t \in (0, 1), \\ u(0) = 0, u'(1) + \sqrt{\lambda} u(1) = 0, \end{cases} \quad (1)$$

with boundary conditions having parameter in two cases  $f(0)=0$  and  $f(0)>0$  by using upper and lower solution method, where  $\lambda > 0$  is a parameter,  $f \in C^2([0, \infty), \mathbb{R})$  is monotonically increasing and  $\lim_{u \rightarrow 1^-} \frac{f(u)}{1-u} = 0, h \in C^1([0, 1], (0, \infty))$  is a nonincreasing function and  $h(t) > 1$ .

From the conclusion of Ref. [15], it follows that problem (1) has a positive solution  $u$  if and only if the problem

$$\begin{cases} -u''(t) = \lambda(1 - (u'(t))^2)^{\frac{3}{2}} h(t)f(u(t)), t \in (0, 1), \\ u(0) = 0, u'(1) + \sqrt{\lambda} u(1) = 0 \end{cases} \quad (2)$$

has a positive solution, and  $|u'| < 1, \|u\|_\infty < 1$ .

Let  $\phi: (-1, 1) \rightarrow \mathbb{R}$  is monotone increasing homeomorphism defined by  $\phi(s) = \frac{s}{\sqrt{1-s^2}}$ , then  $\phi(0) = 0$  and

$\phi^{-1}(s) = \frac{s}{\sqrt{1+s^2}}$  is also monotone increasing homeomorphism and bounded. Let us consider the eigenvalue problem

$$\begin{cases} -u''(t) = \lambda u(t), t \in (0, 1), \\ u(0) = 0, u'(1) + \sqrt{\lambda} u(1) = 0. \end{cases} \quad (3)$$

Based on the results of Ref. [6], the problem (3) has principal eigenvalue  $\lambda_1 > 0$ , and the eigencurve  $B_1(\lambda)$  is Lipschitz continuous, strictly decreasing and convex. Further  $\lim_{\lambda \rightarrow \infty} \lambda_1 = \lambda_{1D}$ , where  $\lambda_{1D} > 0$  is the principal eigenvalue of

$$\begin{cases} -u''(t) = \lambda_{1D} u(t), t \in (0, 1), \\ u(0) = 0 = u(1). \end{cases}$$

Throughout the paper we will assume that  $f$  satisfies:

(C<sub>1</sub>)  $f'(0)=1$ ; (C<sub>2</sub>)  $f''(s)<0, s \in [0, \sigma^*)$  for some  $\sigma^*>0$ .

Let  $a>0, b>0$  and  $M^*>0$ . Define

$$L(a, b) = \frac{4b / (\tau(\frac{1}{4})f(b))}{\min \left\{ \frac{a}{\|v_h\|_\infty f(a)}, \frac{8M^*}{f(b)} \right\}}$$

where  $v_h$  is the solution of the problem

$$\begin{cases} -v_h''(t) = h(t), t \in (0, 1), \\ v_h(0) = 0, v_h'(1) + \sqrt{\lambda} v_h'(1) = 0. \end{cases} \quad (4)$$

First, we state the main results for the case:  $f(0)=0$ .

**Theorem 1** Assume (C<sub>1</sub>) and  $f(0)=0$  hold. Then problem (1) has no positive solution for  $\lambda \approx 0$  and has at least one positive solution for  $\lambda > \lambda_1$ . Further, if there exist  $a > 0, b > 0, M^* > 0$  such that  $a \in (0, b), M^* > 2b, L(a, b) < 1$  and  $\frac{4b}{\tau(\frac{1}{4})f(b)} > \lambda_1$ . Then problem (1) has at least three positive

solutions for  $\lambda \in \left[ \frac{4b}{\tau(\frac{1}{4})f(b)}, \min \left\{ \frac{a}{\|v_h\|_\infty f(a)}, \frac{8M^*}{f(b)} \right\} \right)$ .

**Theorem 2** Assume (C<sub>1</sub>), (C<sub>2</sub>) and  $f(0)=0$  hold.

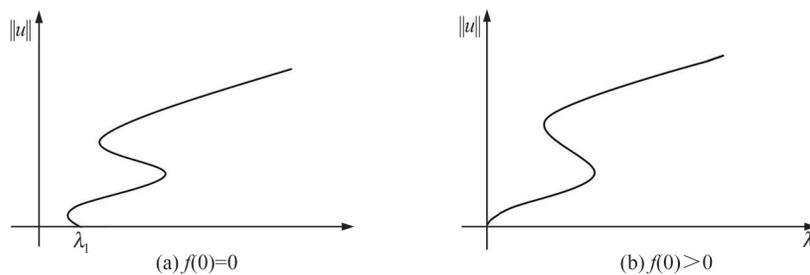


Fig. 1 Bifurcation diagram for problem (1)

### 1 Preliminaries

Let  $C_0^1[0, 1] = \{u \in C^1[0, 1] | u(0)=0\}$ , it is not hard to verify that  $C_0^1[0, 1]$  endowed with the norm  $\|u\| = \|u\|_\infty + \|u'\|_\infty$  is Banach space.

We denote by  $P, Q: C[0, 1] \rightarrow C[0, 1]$ , and the continuous projects defined by  $Pu(t) = u'(0), Qu(t) = \int_0^1 u(t)dt, t \in (0, 1)$  and define the continuous linear operator  $H: C[0, 1] \rightarrow C_0^1[0, 1], Hu(t) = \int_0^t u(s)ds, t \in (0, 1)$ . Integration of both sides of the equation in problem (1) from 0 to  $t$  implies that

Then problem (1) has least one positive solution  $u_\lambda$  for  $\lambda > \lambda_1$  such that  $\|u_\lambda\|_\infty \rightarrow 0$ .

Next, we state the main results for case:  $f(0)>0$ .

**Theorem 3** Assume  $f(0)>0$ . Then problem (1) as at least one positive solution for  $\lambda > 0$ . Further, if there exist  $a > 0, b > 0$  and  $M^* > 0$  such that  $a \in (0, b), M^* > 2b$  and  $L(a, b) < 1$ . Then (1) has at least three positive solu-

tions for  $\lambda \in \left[ \frac{4b}{\tau(\frac{1}{4})f(b)}, \min \left\{ \frac{a}{\|v_h\|_\infty f(a)}, \frac{8M^*}{f(b)} \right\} \right)$ .

Note that the connected component of positive solutions of (1) is the  $\Sigma$  shaped under the assumptions of Theorem 1 or Theorem 3. Figure 1(a) illustrated the main results of Theorem 1, Fig. 1 (b) illustrated the main results of Theorem 3. The rest of the paper is organized as follows. In Section 1, we introduce some lemmas needed to prove the main theorems. In Section 2, we use the time-mapping method to prove that problem (1) has at least one positive solution when  $f(u)=u, h(t)=1$ , which will help to construct the subsolution of (1). In Section 3, we give proofs of the main results. In Section 4, we show some examples.

$$\frac{u'(t)}{\sqrt{1-(u'(t))^2}} = \frac{u'(0)}{\sqrt{1-(u'(0))^2}} - \lambda \int_0^t h(s)f(u(s))ds,$$

i.e.

$$\phi(u'(t)) = \phi(u'(0)) - \lambda \int_0^t h(s)f(u(s))ds.$$

Applying  $\phi^{-1}$  to the above equation, we get that  $u'(t) = \phi^{-1}[\phi(u'(0)) - \lambda \int_0^t h(s)f(u(s))ds]$ . Furthermore, we integrate the above equation from 0 to  $t$ , it follows that

$$u(t) = \int_0^t \phi^{-1}[\phi(u'(0)) - \lambda \int_0^\tau h(s)f(u(s))ds] ds. \quad (5)$$

Combining  $u'(1) + \sqrt{\lambda} u(1) = 0$ , we obtain that  $u'(0)$  satisfies

$$\begin{aligned} &\phi^{-1}[\phi(u'(0)) - \lambda \int_0^t h(s)f(u(s))ds] + \sqrt{\lambda} \int_0^1 \phi^{-1}[\phi(u'(0)) \\ &\quad - \lambda \int_0^t h(s)f(u(\tau))d\tau] ds = 0, \end{aligned} \tag{6}$$

which yields that

$$\begin{aligned} &\phi^{-1}[\phi(Pu(t)) - \lambda Q(h(t)f(u(t)))] \\ &\quad + \sqrt{\lambda} Q[\phi^{-1}(\phi(Pu(t)) - \lambda H(h(t)f(u(t))))] = 0. \end{aligned} \tag{7}$$

**Lemma 1** For  $h \in C[0, 1]$ , there is a unique  $\gamma := u'(0)$  such that  $\phi^{-1}[\phi(\gamma) - \lambda Q(h(t)f(u(t)))] + \sqrt{\lambda} Q[\phi^{-1}(\phi(\gamma) - \lambda H(h(t)f(u(t))))] = 0$  and  $\gamma$  is continuous.

**Proof** For any given  $\lambda > 0$ , define the function  $g: (-1, 1) \rightarrow \mathbb{R}$  as follows:

$$\begin{aligned} g(\gamma) &= \phi^{-1}[\phi(\gamma) - \lambda Q(h(t)f(u(t)))] \\ &\quad + \sqrt{\lambda} Q[\phi^{-1}(\phi(\gamma) - \lambda H(h(t)f(u(t))))] \end{aligned}$$

There are  $\gamma_1, \gamma_2 \in (-1, 1)$  such that

$$\begin{aligned} g(\gamma_1) &= \min_{\gamma \in (0,1)} g(\gamma) = -1 - \sqrt{\lambda} < 0, \\ g(\gamma_2) &= \min_{\gamma \in (0,1)} g(\gamma) = 1 + \sqrt{\lambda} > 0, \end{aligned}$$

then there exists  $\gamma \in (\gamma_1, \gamma_2)$  such that  $g(\gamma) = 0$ .

Next we prove uniqueness of  $\gamma$ . Let  $\gamma_*, \gamma^* \in (-1, 1)$  such that  $g(\gamma_*) = g(\gamma^*) = 0$ , hence there exists  $t_0 \in (0, 1)$  such that

$$\begin{aligned} &\phi^{-1}[\phi(\gamma_*) - \lambda Q(h(t_0)f(u(t_0)))] \\ &\quad + \sqrt{\lambda} Q[\phi^{-1}(\phi(\gamma_*) - \lambda H(h(t_0)f(u(t_0))))] \\ &= \phi^{-1}[\phi(\gamma^*) - \lambda Q(h(t_0)f(u(t_0)))] \\ &\quad + \sqrt{\lambda} Q[\phi^{-1}(\phi(\gamma^*) - \lambda H(h(t_0)f(u(t_0))))]. \end{aligned}$$

Since  $\phi^{-1}$  and  $Q$  is bijection,  $\gamma_* = \gamma^*$ . Finally, from the continuity of  $h, \phi$  and  $\phi^{-1}$ , we can conclude that  $\gamma$  is continuous.

**Lemma 2**  $u \in C_0^1[0, 1]$  being a solution to problem (1) is equivalent to  $u \in C_0^1[0, 1]$  being a fixed point of

$$A_\gamma(u) := H\phi^{-1}[\phi(Pu) - \lambda H(hf(u))].$$

**Proof** It follows from Lemma 1 that there exists a unique  $u'(0)$  such that (7) holds, so the operator equation of problem (1) is  $A_\gamma(u) := H\phi^{-1}[\phi(Pu) - \lambda H(hf(u))]$ . It is obvious that  $\phi: (-1, 1) \rightarrow \mathbb{R}$ , hence its inverse mapping  $\phi^{-1}$  is a bounded operator, and it follows from (5) that  $\|u\|_\infty < 1$ .

Next, we introduce definitions of a (strict) subsolution and a (strict) supersolution of problem (1) and establish the upper and lower solution theorem that is used to prove existence and multiplicity results of (1).

By a supersolution of the problem (1) we define  $\alpha \in C^2(0, 1) \cap C^1[0, 1]$  that satisfies

$$\begin{cases} -\left(\frac{\alpha'(t)}{\sqrt{1-(\alpha'(t))^2}}\right)' \leq \lambda h(t)f(\alpha(t)), t \in (0, 1), \\ \alpha(0) \leq 0, \alpha'(1) + \sqrt{\lambda} \alpha(1) \leq 0. \end{cases} \tag{8}$$

By a subsolution of the problem (1), we define  $\beta \in C^2(0, 1) \cap C^1[0, 1]$  that satisfies

$$\begin{cases} -\left(\frac{\beta'(t)}{\sqrt{1-(\beta'(t))^2}}\right)' \geq \lambda h(t)f(\beta(t)), t \in (0, 1), \\ \beta(0) \geq 0, \beta'(1) + \sqrt{\lambda} \beta(1) \geq 0. \end{cases} \tag{9}$$

By a strict subsolution (supersolution) of (1) we mean a subsolution (supersolution) which is not a solution of (1).

**Lemma 3** Let  $M > 0$  and  $v_1, v_2 \in C^1[0, 1]$  such that  $\phi(v'_i) \in C^1[0, 1] (i = 1, 2)$ . If

$$-(\phi(v'_1(t)))' + Mv_1(t) \leq -(\phi(v'_2(t)))' + Mv_2(t), \forall t \in (0, 1), \tag{10}$$

and  $v_1(0) = 0, v_2(0) = 0$ , then  $v_1(t) \leq v_2(t)$ .

**Proof** Suppose on the contrary that there exists  $t_* \in (0, 1)$  such that  $\max_{t \in (0,1)} (v_1(t) - v_2(t)) = v_1(t_*) - v_2(t_*) > 0$ , then  $v'_1(t_*) = v'_2(t_*)$ , and there exists a sequence  $\{t_k\} \rightarrow t_*$  on  $(0, t_*)$  such that  $v'_1(t_k) - v'_2(t_k) \geq 0$ . The fact that  $\phi$  is monotone increasing homeomorphism implies that  $\phi(v'_1(t_k)) - \phi(v'_1(t_*)) \geq \phi(v'_2(t_k)) - \phi(v'_2(t_*))$ .

By the definition of the derivative we get that

$$(\phi(v'_1(t_k)))'_{t=t_k} \leq (\phi(v'_1(t_*)))'_{t=t_*}. \tag{11}$$

This together with (10) and (11) concludes that  $0 < M(v_1(t_*) - v_2(t_*)) \leq (\phi(v'_1(t)))'_{t=t_*} - (\phi(v'_2(t)))'_{t=t_*} \leq 0$ , which is contradictory. Therefore,  $v_1(t) \leq v_2(t)$  for all  $t \in (0, 1)$ .

**Corollary 1** Let  $v_1, v_2 \in C^1[0, 1]$  such that  $\phi(v_i) \in C^1[0, 1] (i = 1, 2)$ . If  $-(\phi(v'_1(t)))' < -(\phi(v'_2(t)))'$  for any  $t \in (0, 1)$ , then  $v_1(t) < v_2(t)$ .

**Lemma 4** Let  $\alpha$  and  $\beta$  be a subsolution and a supersolution of (1), respectively, such that  $\alpha \leq \beta$ , then (1) has a solution  $u \in C^2(0, 1) \cap C^1[0, 1]$  such that  $u \in [\alpha, \beta]$ .

**Proof** Let  $\chi: \mathbb{R} \rightarrow \mathbb{R}$  be the continuous function which is defined by

$$\chi(u) = \begin{cases} \alpha(t), & u(t) < \alpha(t), \\ u(t), & \alpha(t) \leq u(t) \leq \beta(t), \\ \beta(t), & u(t) > \beta(t), \end{cases}$$

and define  $F: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  by  $F(t, u) = -\lambda h(t)f(\chi(u))$ . Obviously,  $F$  is continuous and bounded.

Next we consider the auxiliary problem

$$\begin{cases} (\phi(u'(t)))' = F(t, u(t)) + u(t) - \chi(u(t)), t \in (0, 1), \\ u(0) = 0, u'(1) + \sqrt{\lambda} u(1) = 0. \end{cases} \tag{12}$$

First, the problem (12) has at least one solution  $u$  by the Schauder fixed theorem. Next, we only show that  $\alpha(t) \leq$

$u(t) \leq \beta(t)$ ,  $t \in (0, 1)$ , so  $u$  is a solution of (1).

Suppose on the contrary that there exists  $\bar{t} \in (0, 1)$  such that  $\max_{t \in (0, 1)} [\alpha(t) - u(t)] = \alpha(\bar{t}) - u(\bar{t}) > 0$ . Because  $\alpha'(\bar{t}) - u'(\bar{t}) = 0$ , there exist two sequence  $\{t_k\} \in [\bar{t} - \varepsilon, \bar{t})$  and  $\{\tilde{t}_k\} \in [\bar{t}, \bar{t} + \varepsilon)$  such that  $\alpha'(t_k) - u'(t_k) \geq 0$  and  $\alpha'(\tilde{t}_k) - u'(\tilde{t}_k) \leq 0$ . Without loss of generality, it follows from  $\alpha'(t_k) - u'(t_k) \geq 0$  that  $\alpha'(t_k) \geq u'(t_k)$ . By  $\phi$  is an increasing homeomorphism, we get that  $\phi(\alpha'(t_k)) \geq \phi(u'(t_k))$ .

Since  $\phi(\alpha'(\bar{t})) = \phi(u'(\bar{t}))$ , then

$$\frac{\phi(\alpha'(t_k)) - \phi(\alpha'(\bar{t}))}{t_k - \bar{t}} \leq \frac{\phi(u'(t_k)) - \phi(u'(\bar{t}))}{t_k - \bar{t}} \quad (t_k < \bar{t}),$$

which implies  $(\phi(\alpha'(\bar{t})))' \leq (\phi(u'(\bar{t})))'$ .

Moreover,  $\alpha$  is a subsolution of (1), it yields that

$$\begin{aligned} (\phi(\alpha'(\bar{t})))' &\leq (\phi(u'(\bar{t})))' = F(\bar{t}, u(\bar{t})) + u(\bar{t}) + \alpha(\bar{t}) \\ &< F(\bar{t}, u(\bar{t})) = -\lambda h(t) f(\chi(u(\bar{t}))) = -\lambda h(t) f(\alpha(\bar{t})) \leq (\phi(\alpha'(\bar{t})))'. \end{aligned}$$

This is a contradiction, thus  $\alpha(t) \leq u(t)$ . In addition, by the similar arguments, it follows that  $u(t) \leq \beta(t)$ . Therefore  $u \in [\alpha, \beta]$  is a solution of problem (1).

**Lemma 5** Let  $\alpha_1$  and  $\beta_1$  be a subsolution and supersolution of (1) respectively such that  $\alpha_1 \leq \beta_1$ . Let  $\alpha_2$  and  $\beta_2$  be a strict subsolution and a strict supersolution of (1) respectively satisfying  $\alpha_2, \beta_2 \in [\alpha_1, \beta_1]$  and  $\alpha_2 < \beta_2$ . Then (1) has at least three solutions  $u_1, u_2$  and  $u_3$ , where  $u_1 \in [\alpha_1, \beta_2]$ ,  $u_2 \in [\alpha_2, \beta_1]$  and  $u_3 \in [\alpha_1, \beta_1] \setminus ([\alpha_1, \beta_2] \cup [\alpha_2, \beta_1])$ .

**Proof** Let  $e \in C^2[0, 1]$  denote the the unique positive solution of

$$\begin{cases} -\left(\frac{e'(t)}{\sqrt{1-(e'(t))^2}}\right)' = 1, t \in (0, 1), \\ e(0) = e(1) = 0. \end{cases}$$

Let  $I = [\alpha_1, \beta_1]$ ,  $I_1 = [\alpha_1, \beta_2]$ ,  $I_3 = [\alpha_2, \beta_1]$ ,  $C_e[0, 1] = \{u \in C[0, 1] \mid -te \leq u \leq te\}$ . It is not difficult to verify that  $C_e[0, 1]$  endowed with the norm  $\|u\|_e = \inf\{t > 0 \mid -te \leq u \leq te\}$  is Banach space, then  $I, I_1$  and  $I_2$  are non-empty closed convex subsets of Banach space  $C_e[0, 1]$ . In the following, we will prove that  $A_f(u): C_0^1[0, 1] \rightarrow C_0^1[0, 1]$  is completely continuous and strongly increasing.

In fact,  $I_1$  and  $I_2$  are disjoint subsets of  $I$ . The map  $A_f(u)$  is restricted to  $I$ . Since  $A_f(u)$  is increasing and  $\alpha_1, \beta_2$  are subsolution and supersolution of (1) respectively ( $\alpha_1 \leq \beta_2$ ), we have that  $\alpha_1 \leq A_f(\alpha_1) \leq A_f(\beta_2) \leq \beta_2$ , i. e.  $A_f(u) \subset I$ . Similarly,  $A_f(I_k) \subset I_k, k=1, 2$ . Since  $\beta_2$  is a strict supersolution of (2), we have that  $A_f(\beta_2) < \beta_2$ . By Ref. [21], Corollary 6.2,  $A_f(u)$  has a maximal fixed point in  $u_1 \in I_1$  and  $\alpha_1 \leq u_1 < \beta_2$ . Similarly,  $A_f(u)$  has a minimal

fixed point in  $u_2 \in I_2$  and  $\alpha_2 < u_2 \leq \beta_1$ , because  $\alpha_2$  is a strict subsolution of (1).

Notice that

$$-(\phi(u_1'(t)))' + Mu_1(t) \leq -(\phi(\beta_2'(t)))' + M\beta_2(t) (M > 0),$$

it follows from Lemma 3 that  $u_1 \leq \beta_2$ , hence there exists a constant  $c_1 > 0$  such that  $\beta_2 - u_1 \geq c_1 e(t)$ . Similarly there exists a constant  $c_2 > 0$  such that  $u_2 - \alpha_2 \geq c_2 e(t)$ . Define

$$J_k = I \cap \left\{ z \in C_e[0, 1] \mid \|z - u_k\|_e < t_k, k=1, 2 \right\},$$

then  $J_k \subset I_k$  is open set,  $k=1, 2$ . Hence  $I_k$  has non-empty interior. Let  $J_k$  be the largest open set in  $I_k$ , which contain  $u_k$  such that  $A_f(u)$  has no fixed point in  $I_k \setminus J_k$ . Finally by Ref. [22], Lemma 3.8,  $A_f(u)$  has a third fixed point  $u_3 \in I \setminus (I_1 \cup I_2)$ . Therefore, problem (1) has at least three solutions  $u_1 \in [\alpha_1, \beta_2]$ ,  $u_2 \in [\alpha_2, \beta_1]$  and  $u_3 \in [\alpha_1, \beta_1] \setminus ([\alpha_1, \beta_2] \cup [\alpha_2, \beta_1])$ .

## 2 The Autonomous Case of $f(u) = u, h(t) = 1$

We consider the quasilinear problem

$$\begin{cases} -\left(\frac{u'(t)}{\sqrt{1-(u'(t))^2}}\right)' = \lambda u(t), t \in (0, 1), \\ u(0) = 0, u'(1) + \sqrt{\lambda} u(1) = 0, \end{cases} \quad (13)$$

where  $\lambda > 0$  is a parameter.

Multiplying the differential equation in (13) by  $u'(t)$  and integrating from 0 to  $t$ , we get that  $-(1 - (u'(t))^2)^{-\frac{1}{2}} = \lambda F(u(t)) + C$ , where  $F(s) = \int_0^s z dz$ . From Ref. [8], Lemma 2.3, there exists  $t_0 \in (0, 1)$  such that  $u(t_0) = \rho = \|u\|_\infty$ . Now we choose  $t = t_0$  in (13) and yield

$$u'(t) = \sqrt{1 - (1 + \lambda(F(\rho) - F(u)))^2}, t \in (0, t_0), \quad (14)$$

$$u'(t) = -\sqrt{1 - (1 + \lambda(F(\rho) - F(u)))^2}, t \in (t_0, 1). \quad (15)$$

Let  $u(1) = m > 0$ , further integration from 0 to  $t_0$  for (14) and integration from  $t_0$  to 1 for (15), it follows that

$$\int_0^{\rho} \frac{du}{\sqrt{1 - (1 + \lambda(F(\rho) - F(u)))^2}} = t_0, \quad (16)$$

$$\int_{\rho}^m \frac{du}{\sqrt{1 - (1 + \lambda(F(\rho) - F(u)))^2}} = t_0 - 1. \quad (17)$$

We obtain by combining (16) and (17) that

$$\int_0^{\rho} \frac{du}{\sqrt{1 - (1 + \lambda(F(\rho) - F(u)))^2}} + \int_{\rho}^m \frac{du}{\sqrt{1 - (1 + \lambda(F(\rho) - F(u)))^2}} = 1. \quad (18)$$

Finally, from boundary value  $u'(1) + \sqrt{\lambda} u(1) = 0$  and (15), we can get

$$G_\lambda(m) := (1 + \lambda(F(\rho) - F(m)))^{-2} + \lambda m^2 - 1 = 0. \tag{19}$$

Let  $\lambda$  be a fixed value and  $\rho \in (0, 1)$ . By (19), it is easy to compute that

$$\frac{\partial G_\lambda}{\partial m} = 2\lambda m \left[ 1 + \frac{1}{\left(1 + \frac{\lambda}{2}(\rho^2 - m^2)\right)^3} \right] > 0, m \in (0, \rho), \tag{20}$$

further  $G_\lambda(0) = \left(1 + \frac{\lambda}{2}\rho^2\right)^{-2} - 1 < 0$ ,  $G_\lambda(\rho) = \lambda\rho^2 > 0$ , so it follows from the intermediate value theorem that there exists  $\tilde{m} = \tilde{m}(\rho) \in (0, \rho)$  such that (19) holds.

**Lemma 6** For any fixed  $\rho \in (0, 1)$ ,  $\lim_{\rho \rightarrow 0} \frac{\tilde{m}}{\rho} = \frac{\sqrt{2}}{2}$  for  $\tilde{m} \in (0, \rho)$ .

**Proof** By (19), we get that  $G_\lambda(\tilde{m}) = 0$ . It is easy to compute that

$$\frac{\partial G_\lambda(\tilde{m})}{\partial \rho} = \frac{-2\lambda\rho}{\left(1 + \frac{\lambda}{2}(\rho^2 - \tilde{m}^2)\right)^3}, \frac{\partial G_\lambda(\tilde{m})}{\partial \tilde{m}} = 2\lambda\tilde{m} \left[ 1 + \frac{\lambda}{2}(\rho^2 - \tilde{m}^2)^{-3} \right].$$

Thus  $\lim_{\rho \rightarrow 0} \left(\frac{\tilde{m}}{\rho}\right)^2 = \lim_{\rho \rightarrow 0} \tilde{m}(\rho) = \lim_{\rho \rightarrow 0} \frac{1}{\left[1 + \frac{\lambda}{2}(\rho^2 - \tilde{m}^2)\right]^3} = \frac{1}{2}$ , i.e.  $\lim_{\rho \rightarrow 0} \frac{\tilde{m}}{\rho} = \frac{\sqrt{2}}{2}$ .

**Definition 1** For any given  $\rho \in (0, 1)$ , we define a map

$$T_{\lambda, \tilde{m}}(\rho) = \int_0^\rho \frac{du}{\sqrt{1 - (1 + \lambda(F(\rho) - F(u)))^{-2}}} + \int_{\tilde{m}}^\rho \frac{du}{\sqrt{1 - (1 + \lambda(F(\rho) - F(u)))^{-2}}}, \tag{21}$$

called the time map of problem (13).

In fact, the existence of a solution to problem (13) is equivalent to the existence of a solution to  $T_{\lambda, \tilde{m}}(\rho) = 1$ , see Refs. [19, 23].

**Theorem 4** For any given  $\rho \in (0, 1)$ , there exists  $\lambda_1 = \lambda_1(\rho, \tilde{m})$  satisfying (18) and (19) such that problem (13) has at least one positive solution for any  $\lambda > \lambda_1$ .

**Proof** Clearly, for any fixed  $\lambda > 0$ , there exists  $\tilde{m} = \tilde{m}(\rho)$  such that  $G_\lambda(\tilde{m}) = 0$ . Now, let  $m = \tilde{m}$  in (21), we show that

$$T_{\lambda, \tilde{m}}(\rho) = \int_0^\rho \frac{du}{\sqrt{1 - (1 + \lambda(F(\rho) - F(u)))^{-2}}} + \int_{\tilde{m}}^\rho \frac{du}{\sqrt{1 - (1 + \lambda(F(\rho) - F(u)))^{-2}}} = 1. \tag{22}$$

It is easy to compute that  $\frac{\partial T_{\lambda, \tilde{m}}(\rho)}{\partial \rho} = -\tilde{m}'(\rho) \frac{1}{\sqrt{1 - (1 + \lambda(F(\rho) - F(\tilde{m})))^{-2}}} < 0$ ,  $\tilde{m} \in (0, \rho)$ , and  $T_{\lambda, \tilde{m}}(0) = 0 < 1$ ,

$$T_{\lambda, \tilde{m}}(1) = \int_0^1 \frac{du}{\sqrt{1 - (1 + \lambda(F(1) - F(u)))^{-2}}} + \int_{\tilde{m}(1)}^1 \frac{du}{\sqrt{1 - (1 + \lambda(F(1) - F(u)))^{-2}}} > 1.$$

Therefore, there exist  $\rho \in (0, 1)$  and  $\tilde{m} = \tilde{m}(\rho)$ , such that  $T_{\lambda, \tilde{m}}(\rho) = 1$  holds. Then problem (13) has at least one positive solution.

Finally, we discuss the range of value of  $\lambda$  when a positive solution exists for problem (13). The fact together with (18) concludes that

$$\begin{aligned} 1 &= \lim_{\rho \rightarrow 0^+} \left( \int_0^\rho \frac{du}{\sqrt{1 - (1 + \lambda(F(\rho) - F(u)))^{-2}}} + \int_{\tilde{m}}^\rho \frac{du}{\sqrt{1 - (1 + \lambda(F(\rho) - F(u)))^{-2}}} \right) \\ &= \lim_{\rho \rightarrow 0^+} \left( \int_0^\rho \frac{1 + \lambda(F(\rho) - F(u))}{\sqrt{(1 + \lambda(F(\rho) - F(u)))^2 - 1}} du + \int_{\tilde{m}}^\rho \frac{1 + \lambda(F(\rho) - F(u))}{\sqrt{(1 + \lambda(F(\rho) - F(u)))^2 - 1}} du \right) \\ &= \lim_{\rho \rightarrow 0^+} \left( \int_0^\rho \frac{1 + \frac{\lambda}{2}(\rho^2 - u^2)}{\sqrt{\frac{\lambda^2}{4}(\rho^2 - u^2)^2 + \lambda(\rho^2 - u^2)}} du + \int_{\tilde{m}}^\rho \frac{1 + \frac{\lambda}{2}(\rho^2 - u^2)}{\sqrt{\frac{\lambda^2}{4}(\rho^2 - u^2)^2 + \lambda(\rho^2 - u^2)}} du \right) \end{aligned}$$

$$\begin{aligned}
 &= \lim_{\rho \rightarrow 0^+} \left( \int_0^1 \frac{(1 + \frac{\lambda}{2}(\rho^2 - (\rho\tau)^2))\rho}{\sqrt{\frac{\lambda^2}{4}(\rho^2 - (\rho\tau)^2)^2 + \lambda(\rho^2 - (\rho\tau)^2)}} d\tau + \int_{\frac{\tilde{m}}{\rho}}^1 \frac{(1 + \frac{\lambda}{2}(\rho^2 - (\rho\tau)^2))\rho}{\sqrt{\frac{\lambda^2}{4}(\rho^2 - (\rho\tau)^2)^2 + \lambda(\rho^2 - (\rho\tau)^2)}} d\tau \right) \\
 &= \lim_{\rho \rightarrow 0^+} \left( \int_0^1 \frac{1 + \frac{\lambda}{2}(\rho^2 - (\rho\tau)^2)}{\sqrt{\frac{\lambda^2}{4}(\rho - \rho\tau^2)^2 + \lambda(1 - \tau^2)}} d\tau + \int_{\frac{\tilde{m}}{\rho}}^1 \frac{1 + \frac{\lambda}{2}(\rho^2 - (\rho\tau)^2)}{\sqrt{\frac{\lambda^2}{4}(\rho - \rho\tau^2)^2 + \lambda(1 - \tau^2)}} d\tau \right) \\
 &= \frac{1}{\sqrt{\lambda}} \left( \int_0^1 \frac{1}{\sqrt{1 - \tau^2}} d\tau + \int_{\frac{\sqrt{2}}{2}}^1 \frac{1}{\sqrt{1 - \tau^2}} d\tau \right) = \frac{1}{\sqrt{\lambda}} \frac{3\pi}{4}.
 \end{aligned}$$

Therefore,  $\lim_{\rho \rightarrow 0^+} \lambda(\rho, \tilde{m}) = \frac{9}{16} \pi^2 := \lambda_1$ .

Next, we show that  $\lim_{\rho \rightarrow 1^-} \lambda(\rho, \tilde{m}) = \infty$ . From the definition of  $\phi(u)$ , we get that  $\lim_{\rho \rightarrow 1^-} \frac{u}{\phi(u)} = 0$ . Hence, there exists  $\delta \in (0, 1)$  for any  $\varepsilon > 0$  such that  $u \leq \varepsilon\phi(u)$ ,  $\forall 1 - \delta < u < 1$  and it follows that

$$F(\rho) - F(u) = \int_u^\rho s ds < \varepsilon \int_u^\rho \frac{s}{\sqrt{1 - s^2}} ds = \frac{\varepsilon(\rho^2 - u^2)}{\sqrt{1 - u^2} + \sqrt{1 - \rho^2}}.$$

This together with (18) implies that

$$\begin{aligned}
 1 &= \int_0^\rho \frac{du}{\sqrt{1 - (1 + \lambda(F(\rho) - F(u)))^2}} + \int_{\tilde{m}}^\rho \frac{du}{\sqrt{1 - (1 + \lambda(F(\rho) - F(u)))^2}} \\
 &\geq \int_0^\rho \frac{du}{\sqrt{1 - [1 + \lambda(F(\rho) - F(u))]^2}} \\
 &\geq \int_0^\rho \frac{du}{\sqrt{1 - [1 + \lambda\varepsilon(\rho^2 - u^2)](\sqrt{1 - u^2} + \sqrt{1 - \rho^2})^2}} \\
 &= \int_0^1 \frac{1 + \lambda\varepsilon\rho^2(1 - \tau^2)(\sqrt{1 - (\rho\tau)^2} + \sqrt{1 - \rho^2})}{\sqrt{\lambda\varepsilon \left[ \frac{2(1 - \tau^2)}{\sqrt{1 - (\rho\tau)^2} + \sqrt{1 - \rho^2}} + \frac{\lambda\varepsilon\rho^2(1 - \tau^2)^2}{(\sqrt{1 - (\rho\tau)^2} + \sqrt{1 - \rho^2})^2} \right]}} d\tau.
 \end{aligned}$$

Therefore, we yield that for  $\varepsilon > 0$  small enough as  $\rho \rightarrow 1^-$ ,  $\lim_{\rho \rightarrow 1^-} \sqrt{\lambda(\rho, \tilde{m})} \geq \lim_{\rho \rightarrow 1^-} \frac{1}{\sqrt{\varepsilon}} \int_0^1 \frac{1}{\sqrt{1 - \tau^2}} d\tau = \infty$ , i.e.  $\lim_{\rho \rightarrow 1^-} \lambda(\rho, \tilde{m}) = \infty$ .

In consequence, the problem (13) has at least one positive solution for any  $\lambda > \lambda_1$ .

**Remark 1** In the case of  $h(t) = 1, f(u) = u$ , the principal eigenvalue of problem

$$\begin{cases} -u''(t) = \lambda u(t), t \in (0, 1), \\ u(0) = 0, u'(1) + \sqrt{\lambda} u(1) = 0 \end{cases}$$

has eigenvalue  $\lambda_1 = \frac{9}{16} \pi^2$ , and the corresponding eigenfunction is  $B_1(\lambda) = \sin \frac{3\pi}{4} t, t \in (0, 1)$ .

### 3 The Proof of Main Results

**Proof of Theorem 1** We first show the nonexistence for  $\lambda \approx 0$ . Let  $\sigma_\lambda$  be the principal eigenvalue and  $\omega_\lambda > 0$  be the corresponding normalized eigenfunction of

$$\begin{cases} -\omega_\lambda''(t) = (\lambda + \sigma)\omega_\lambda(t), t \in (0, 1), \\ \omega_\lambda(0) = 0, \omega_\lambda'(1) + \sqrt{\lambda} \omega_\lambda(1) = 0. \end{cases}$$

We note that  $\sigma_\lambda > 0$  for  $\lambda < \lambda_1$  and  $\sigma_\lambda < 0$  for  $\lambda > \lambda_1$ , the detail see Ref. [24]. Suppose on the contrary that  $u_\lambda$  be a positive solution of (1) for  $\lambda \approx 0$ . Note that there exists  $k_0 > 0$  such that  $f(s) \leq k_0 s$  for  $s \in [0, \infty)$ , and

$$\begin{aligned}
 0 &= \int_0^1 (u_\lambda'' \omega_\lambda - \omega_\lambda'' u_\lambda) ds = \int_0^1 h(s) [u_\lambda \omega_\lambda (\lambda + \sigma_\lambda) - \lambda f(u_\lambda) \omega_\lambda (1 - (u_\lambda')^2)^{\frac{3}{2}}] ds \\
 &\geq \int_0^1 h(s) [(\lambda + \sigma_\lambda) - \lambda k_0 (1 - (u_\lambda')^2)^{\frac{3}{2}}] u_\lambda \omega_\lambda ds \geq \int_0^1 h(s) [(\lambda + \sigma_\lambda) - \lambda k_0] u_\lambda \omega_\lambda ds.
 \end{aligned}
 \tag{23}$$

Clearly, it is easy to see that  $\sqrt{\lambda} > \lambda$  for  $\lambda \approx 0$ . Thus there exists constant  $m > 0$  (independent of  $\lambda$ ) such that  $0 < m \sqrt{\lambda} - \lambda < \sigma_\lambda$ . Besides, by (23), it follows that  $(\lambda + \sigma_\lambda) - \lambda k_0 \leq 0$ , i.e.  $\frac{\sigma_\lambda}{\lambda} \not\rightarrow \infty$ . This is a contradiction, hence (1) has no positive solution for  $\lambda \approx 0$ .

Next, we show that the existence of positive solution of (1) for  $\lambda > \lambda_1$ . Let  $\beta_1 = m_\lambda e_h$ , where  $e_h$  is solution of

$$\begin{cases}
 -\left(\frac{u'(t)}{\sqrt{1-(u'(t))^2}}\right)' = h(t), t \in (0, 1), \\
 u(0) = 0, u'(1) + \sqrt{\lambda} u(1) = 0
 \end{cases}$$

and  $1 < m_\lambda < \frac{1}{\min\{\|e_h\|_\infty, \|e_h'\|_\infty\}}$ . Then  $1 - m_\lambda \|e_h\|_\infty > 0$  and  $1 - m_\lambda^2 e_h'^2 > 0$ .

Since  $\lim_{u \rightarrow 1} \frac{f(u)}{1-u} = 0$ , for all  $\varepsilon > 0$ , there exists  $0 < \delta < 1$  such that  $\left|\frac{f(u)}{1-u}\right| < \varepsilon$ ,  $1 - \delta < u < 1$ . Furthermore, choosing  $\varepsilon = \frac{m_\lambda}{\lambda(1 - m_\lambda \|e_h\|_\infty)}$ , we have that

$$\frac{f(m_\lambda \|e_h\|_\infty)}{1 - m_\lambda \|e_h\|_\infty} < \frac{m_\lambda}{\lambda(1 - m_\lambda \|e_h\|_\infty)}, \text{ i.e. } \lambda f(m_\lambda \|e_h\|_\infty) < m_\lambda.
 \tag{24}$$

Next, we prove that  $\beta_1$  is an upper solution to problem (1). By  $m_\lambda > 1$  and (24), we conclude that

$$-\left(\frac{\beta_1'}{\sqrt{1-\beta_1'^2}}\right)' = -\frac{\beta_1''}{(1-\beta_1'^2)^{\frac{3}{2}}} = -\frac{m_\lambda e_h''}{(1-m_\lambda^2 e_h'^2)^{\frac{3}{2}}} > -\frac{m_\lambda e_h''}{(1-e_h'^2)^{\frac{3}{2}}} = m_\lambda h(t) \geq \lambda f(m_\lambda e_h) h(t) = \lambda f(\beta_1) h(t).$$

In addition, we get that  $\beta_1(0) = 0$  and  $\beta_1'(1) + \sqrt{\lambda} \beta_1(1) = 0$ , thus  $\beta_1$  is a supersolution.

Let  $\alpha_1 = \varepsilon v$  with  $\varepsilon > 0$ , where  $v$  is positive solution of (13) according to Theorem 4. Since  $f'(0) = 1$ ,  $\lim_{u \rightarrow 0^+} \frac{f(u)}{u} = 1$ .

Therefore, for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $1 - \varepsilon \leq \frac{f(u)}{u} \leq 1 + \varepsilon$ . Hence, for  $\varepsilon \approx 0$ , we have that

$$-\left(\frac{\alpha_1'}{\sqrt{1-\alpha_1'^2}}\right)' = -\left(\frac{\varepsilon v'}{\sqrt{1-\varepsilon^2 v'^2}}\right)' = \frac{\lambda \varepsilon (1-v'^2)^{\frac{3}{2}} v}{(1-\varepsilon^2 v'^2)^{\frac{3}{2}}} \leq \lambda \varepsilon v h(t) \leq \lambda f(\varepsilon v) h(t).$$

Obviously  $\alpha_1(0) = 0$  and  $\alpha_1'(1) + \sqrt{\lambda} \alpha_1(1) = 0$ , thus  $\alpha_1$  is a subsolution of (2). Now choosing  $\varepsilon \approx 0$  to ensure that  $\beta_1 \geq \alpha_1$ . By Lemma 1 there exists a positive solution  $u_\lambda \in [\alpha_1, \beta_1]$  for  $\lambda > \lambda_1$ .

Finally, we establish the multiplicity result of positive solution of (1). Let  $m^*, M^* \in (0, \eta)$  such that  $f$  is strictly increasing on  $[m^*, M^*]$ . We first construct a strict subsolution of the Dirichlet problem

$$\begin{cases}
 -\left(\frac{u'(t)}{\sqrt{1-(u'(t))^2}}\right)' = \lambda h(t) \tilde{f}(u(t)), t \in (0, 1), \\
 u(0) = u(1) = 0.
 \end{cases}
 \tag{25}$$

Define  $\tilde{f} \in C^2([0, \infty), \mathbb{R})$  satisfying

$$\tilde{f}(u) = \begin{cases} \hat{f}(u), & u < m^*, \\ f(u), & u \geq m^*, \end{cases}$$

where  $\hat{f}(u)$  is defined such that the function  $\tilde{f}(u)$  is strictly increasing on  $[m^*, M^*]$  and  $\tilde{f}(u) \leq f(u)$ . For  $t \in [0, \frac{1}{2}]$ , let

$$g(t) = \begin{cases} 1 - (1 - (\frac{t}{\zeta})^\mu)^\delta, & 0 \leq t < \zeta, \\ 1, & \zeta \leq t \leq \frac{1}{2}, \end{cases}$$

and  $g(t) = g(1 - t)$ ,  $t \in [\frac{1}{2}, 1]$ , where  $0 < \mu, \delta < 1$ . By computing, we get that

$$g'(t) = \begin{cases} \frac{\delta\mu}{\zeta} \left(\frac{t}{\zeta}\right)^{\mu-1} \left(1 - \left(\frac{t}{\zeta}\right)^\mu\right)^{\delta-1}, & 0 \leq t < \zeta, \\ 0, & \zeta \leq t \leq \frac{1}{2}. \end{cases}$$

Assume that there exists a constant  $b \in [m^*, M^*)$  such that  $M^* > 2b$ , and define  $z(t) = bg(t)$ . Since  $|g'(t)| < \frac{\delta\mu}{\zeta}$ ,  $|z'(t)| < \frac{b\delta\mu}{\zeta}$ . Define  $\alpha_*$  on  $[\frac{1}{2}, 1]$  to be the solution of

$$\begin{cases} -\left(\frac{\alpha'_*(t)}{\sqrt{1 - (\alpha'_*(t))^2}}\right)' = \lambda \tilde{f}(z(t)), & t \in (0, 1), \\ \alpha_*(0) = \alpha'_*\left(\frac{1}{2}\right) = 0, \end{cases} \tag{26}$$

and extend  $\alpha_*$  on  $[0, \frac{1}{2}]$  such that  $\alpha_*(t) = \alpha_*(1 - t)$ , where  $\alpha_*(t)$  has a maximum value at  $t = \frac{1}{2}$  and is symmetric about  $t = \frac{1}{2}$ , see Ref. [17].

**Claim**  $\alpha_*(t) \in (z(t), M^*)$ ,  $t \in (0, 1)$ .

Based on the claim, it follows that

$$-\alpha''_* = \lambda(1 - (\alpha'_*)^2)^{\frac{3}{2}} \tilde{f}(z) < \lambda(1 - (\alpha'_*)^2)^{\frac{3}{2}} \tilde{f}(\alpha_*) \leq \lambda(1 - (\alpha'_*)^2)^{\frac{3}{2}} h(t) \tilde{f}(\alpha_*(t)).$$

Since  $\alpha_*(0) = \alpha_*(1) = 0$ ,  $\alpha'_*(1) + \sqrt{\lambda} \alpha_*(1) = \alpha'_*(1) \leq 0$ . Therefore,  $\alpha_*$  is a strict subsolution. Now, we prove the claim is true. First, we show that  $\alpha_*(t) > z(t)$ ,  $t \in [0, 1]$ .

Define  $\tau = \tau(\zeta) := \int_{\zeta}^{\frac{1}{2}} (1 - (\alpha'_*(s))^2)^{\frac{3}{2}} ds$ , then  $0 < \tau(\zeta) < 1$ . Recall that (26). Integration from  $t$  to  $\frac{1}{2}$  with  $t \in (0, \frac{1}{2})$  and

noting that  $\alpha'_*\left(\frac{1}{2}\right) = 0$ , we obtain that

$$\alpha'_*(t) = \lambda \int_t^{\frac{1}{2}} (1 - (\alpha'_*(s))^2)^{\frac{3}{2}} \tilde{f}(z(s)) ds \geq \lambda \int_t^{\frac{1}{2}} (1 - (\alpha'_*(s))^2)^{\frac{3}{2}} \tilde{f}(bg(s)) ds = \lambda \tilde{f}(b) \int_t^{\frac{1}{2}} (1 - (\alpha'_*(s))^2)^{\frac{3}{2}} ds = \lambda \tilde{f}(b) \tau(\zeta), \quad t \in [0, \zeta]. \tag{27}$$

Now we choose  $\zeta = \frac{1}{4}$ . Since  $\lambda > \frac{4b}{\tau(\frac{1}{4})f(b)}$ , we can choose  $0 < \mu, \delta < 1$  such that  $\lambda > \frac{4b(\delta\mu)}{\tau(\frac{1}{4})f(b)}$ . Hence for all  $t \in [0, \frac{1}{4}]$ ,

$\alpha'_*(t) > \lambda f(b) \tau\left(\frac{1}{4}\right) > 4b\delta\mu = \frac{b\delta\mu}{\zeta} > z'(t)$ . Clearly,  $z(t) = b$ ,  $z'(t) = 0$  for all  $t \in [\frac{1}{4}, \frac{1}{2}]$ . We obtain that

$$\alpha'_*(t) = \lambda \int_t^{\frac{1}{2}} (1 - (\alpha'_*(s))^2)^{\frac{3}{2}} \tilde{f}(z(s)) ds > 0 = z'(t). \text{ i.e. } \alpha'_*(t) > z'(t) \text{ for all } t \in [0, \frac{1}{2}].$$

Since  $\alpha_*(0) = z(0) = 0$ , we have  $\alpha_*(t) > z(t)$  for all  $t \in [0, \frac{1}{2}]$ . By symmetry of the solution of (26),  $\alpha_*(t) > z(t)$  for all  $t \in [0, 1]$ .

Next, we show that  $\|\alpha_*\|_\infty < M^*$ . Recall that  $\alpha'_*(t) = \lambda \int_t^{\frac{1}{2}} (1 - (\alpha'_*(s))^2)^{\frac{3}{2}} \tilde{f}(z(s)) ds$ ,  $t \in (0, \frac{1}{2})$ . Integrating from 0 to  $t$ , we have

$$\alpha_*(t) = \int_0^t \left( \lambda \int_s^{\frac{1}{2}} (1 - (\alpha'_*(\tau))^2)^{\frac{3}{2}} \tilde{f}(z(\tau)) d\tau \right) ds, \quad t \in (0, \frac{1}{2}).$$

Furthermore,  $\|\alpha_*\|_\infty = \alpha_*\left(\frac{1}{2}\right) \leq \lambda \tilde{f}(b) \int_0^{\frac{1}{2}} \int_s^{\frac{1}{2}} (1 - (\alpha'_*(\tau))^2)^{\frac{3}{2}} d\tau ds = \frac{\lambda \tilde{f}(b)}{8} = \frac{\lambda f(b)}{8}$ . This together with  $\lambda < \frac{8M^*}{f(b)}$  yields

$\|\alpha_*\|_\infty < M^*$ . Hence  $v(t) < \alpha_*(t) < M^*$  for  $t \in (0, 1)$ . Moreover  $\alpha_*$  is a strict subsolution of problem (1).

Let  $\alpha_2$  be the first iteration of  $\alpha_*$ , then  $\alpha_2$  be the solution to the problem

$$\begin{cases} -\left(\frac{\alpha_2'(t)}{\sqrt{1-(\alpha_2'(t))^2}}\right)' = \lambda h(t)f(\alpha_*(t)), t \in (0, 1), \\ \alpha_2(0) = 0, \alpha_2'(1) + \sqrt{\lambda} \alpha_2(1) = 0. \end{cases}$$

Then  $-\left(\frac{\alpha_2'}{\sqrt{1-(\alpha_2')^2}}\right)' = \lambda h(t)f(\alpha_*) > \lambda h(t)f(z) = -\left(\frac{\alpha_*'}{\sqrt{1-(\alpha_*')^2}}\right)'$ . By Corollary 1, we have that  $\alpha_2 > \alpha_*$ . Hence, it is not difficult to verify that  $\alpha_2$  is a strict subsolution of (1).

Finally, we construct a strict supersolution for  $\lambda \in \left(\frac{4b}{\tau\left(\frac{1}{4}\right)f(b)}, \frac{a}{\|v_h\|_\infty f(a)}\right)$ , where  $a \in (0, b)$  is a constant. Let  $\beta_2 := \frac{av_h}{\|v_h\|_\infty}$ , where  $v_h$  is defined by (4). Then

$$-\beta_2'' = \frac{ah(t)}{\|v_h\|_\infty} > \lambda h(t)f(a) > \lambda(1 - (\beta_2')^2)^{\frac{3}{2}}h(t)f(a) > \lambda(1 - (\beta_2')^2)^{\frac{3}{2}}h(t)f(\beta_2).$$

On the other hand,  $\beta_2$  satisfies  $\beta_2(0) = 0$  and  $\beta_2'(1) + \sqrt{\lambda} \beta_2(1) = \frac{a}{\|v_h\|_\infty} [v_h'(1) + \sqrt{\lambda} v_h(1)] = 0$ . Therefore  $\beta_2$  is a strict

supersolution for  $\lambda \in \left(\frac{4b}{\tau\left(\frac{1}{4}\right)f(b)}, \frac{a}{\|v_h\|_\infty f(a)}\right)$ . Further, we can choose  $\varepsilon \approx 0$  and  $0 < a \ll 1$  such that  $\alpha_1 \leq \alpha_2 \leq \beta_1$  and

$\alpha_1 \leq \beta_2 \leq \beta_1$ . Since  $\|\alpha_2\|_\infty \geq b > a = \|\beta_2\|_\infty$ ,  $\alpha_2 \not\leq \beta_2$ . Therefore (1) has at least three positive solutions from Lemma 5.

**Proof of Theorem 2** By  $(C_2)$ , there exists  $A^* > 0$  such that  $f''(s) \leq -A^*$  for  $s \approx 0$ . Let  $\hat{\beta}_1 := \delta_\lambda \omega_\lambda$ , where  $\delta_\lambda = -\frac{2\sigma_\lambda}{\lambda A^* \min_{t \in (0,1)} \omega_\lambda}$ . It follows that  $\delta_\lambda > 0$  and  $\delta_\lambda \rightarrow 0 (\lambda \rightarrow \lambda_1^+)$  from  $\sigma_\lambda < 0$ ,  $\sigma_\lambda \rightarrow 0 (\lambda \rightarrow \lambda_1^+)$  and  $\min_{t \in (0,1)} \omega_\lambda \rightarrow 0 (\lambda \rightarrow \lambda_1^+)$ . By the

Taylor's series  $f(\hat{\beta}_1) = f(0) + f'(0)\hat{\beta}_1 + \frac{f''(\zeta)}{2} \hat{\beta}_1^2 = \hat{\beta}_1 + \frac{f''(\zeta)}{2} \hat{\beta}_1^2$ ,  $\zeta \in [0, \hat{\beta}_1]$ , it concludes that

$$\begin{aligned} -\hat{\beta}_1'' - \lambda(1 - (\hat{\beta}_1')^2)^{\frac{3}{2}}h(t)f(\hat{\beta}_1) &= \delta_\lambda(\lambda + \sigma_\lambda)h(t)\omega_\lambda - \lambda(1 - (\beta_1')^2)^{\frac{3}{2}}h(t)\left[\delta_\lambda\omega_\lambda + \frac{f''(\zeta)}{2}(\delta_\lambda\omega_\lambda)^2\right] \\ &\geq \delta_\lambda(\lambda + \sigma_\lambda)(1 - (\beta_1')^2)^{\frac{3}{2}}h(t)\omega_\lambda - \lambda(1 - (\beta_1')^2)^{\frac{3}{2}}h(t)\left[\delta_\lambda\omega_\lambda + \frac{f''(\zeta)}{2}(\delta_\lambda\omega_\lambda)^2\right] \\ &\geq \delta_\lambda\omega_\lambda(1 - (\beta_1')^2)^{\frac{3}{2}}h(t)\left[\sigma_\lambda + \frac{\lambda A^*}{2} \delta_\lambda \min_{t \in (0,1)} \omega_\lambda\right] = 0. \end{aligned}$$

Therefore,  $\hat{\beta}_1$  is a supersolution of (1) for  $\lambda > \lambda_1$  and  $\|\hat{\beta}_1\|_\infty \rightarrow 0 (\lambda \rightarrow \lambda_1^+)$ . Let  $\alpha_1 = \varepsilon v$  be as in the proof of Theorem 1.

Choosing  $\varepsilon \approx 0$ , we can note that  $\alpha_1 \leq \beta_1$ . By Lemma 2, there exists a positive solution  $u_\lambda \in [\alpha_1, \hat{\beta}_1]$  for  $\lambda > \lambda_1$  and  $\lambda \approx \lambda_1$  such that  $\|u_\lambda\|_\infty \rightarrow 0 (\lambda \rightarrow \lambda_1^+)$ .

**Proof of Theorem 3** First of all, we show the existence of positive solution for  $\lambda > 0$ . Clearly,  $\alpha \equiv 0$  is a strict subsolution of (1). Let  $\beta = m_\lambda e_h$  be as in the proof of Theorem 1, then is a supersolution of (1). Further, by Lemma 1, there exists  $u_\lambda \in [\alpha, \beta]$ . Next, the proof of multiplicity is similar to that of Theorem 1, we omit it.

### 4 Examples

**Example 1** Let us consider the following problem

$$\begin{cases} -\left(\frac{u'(t)}{\sqrt{1-(u'(t))^2}}\right)' = \lambda 3t^{-\frac{1}{2}}f(u(t)), t \in (0, 1), \\ u(0) = 0, u'(1) + \sqrt{\lambda} u(1) = 0, \end{cases} \tag{28}$$

where

$$f(s) = \begin{cases} (1-s)\cos((s-\frac{1}{2})\pi), & s \in [0, \frac{7}{20}), \\ (1-s)\sin((s-1)\pi), & s \in [\frac{7}{20}, 1]. \end{cases}$$

It is obvious that  $f(0)=0$ . Moreover, we note that  $f(s)$  satisfies  $(C_1)$ ,  $(C_2)$  and  $\lim_{s \rightarrow 1^-} \frac{f(s)}{1-s} = 0$ . Let  $s = s_0 = \frac{1}{10}$ , then  $\frac{s_0}{f(s_0)} \approx \frac{10}{27}$ . Let  $a = s_0$  and  $t > 2s_0$ , then we can choose  $b_i \in (s_0, \frac{t}{2})$  such that  $\frac{4b}{t(\frac{1}{4})f(b_i)} < \frac{1}{6}$ . Let  $v_{b_i}$  be as in (4). It is easy to com-

pute that  $v_{b_i}(t) = \frac{6+4\sqrt{\lambda}}{3(1+\sqrt{\lambda})} - \frac{4}{3}t^{\frac{3}{2}}$ ,  $t \in (0, 1)$ . Then we obtain that  $\|v_{b_i}\|_{\infty} \leq 2$ . Hence  $\left(\frac{4b}{t(\frac{1}{4})f(b_i)}\right) / \left(\frac{a}{\|v_{b_i}\|_{\infty}f(a)}\right) < 1$ .

Next, let  $M_i = \frac{t}{2}$ . Then  $\frac{4b_i}{t(\frac{1}{4})f(b_i)} < \min\left\{\frac{a}{\|v_{b_i}\|_{\infty}f(a)}, \frac{8M_i}{f(b_i)}\right\}$ . Therefore, problem (28) has at least three positive solutions for  $\lambda \in \left(\frac{4b_i}{t(\frac{1}{4})f(b_i)}, \min\left\{\frac{a}{\|v_{b_i}\|_{\infty}f(b_i)}, \frac{8M_i^*}{f(b_i)}\right\}\right)$  by Theorem 1.

**Example 2** Let us consider the following problem

$$\begin{cases} -\left(\frac{u'(t)}{\sqrt{1-(u'(t))^2}}\right)' = \lambda 3t^{-\frac{1}{2}}f(u(t)), & t \in (0, 1), \\ u(0) = 0, u'(1) + \sqrt{\lambda}u(1) = 0, \end{cases} \tag{29}$$

where

$$f(s) = \begin{cases} (\frac{1}{200} + s)(1-s)\ln(2-s), & s \in [0, \frac{7}{20}), \\ (\frac{1}{200} + s)(1-s)\ln(e^{1-s} - 1), & s \in [\frac{7}{20}, 1]. \end{cases}$$

It is obvious that  $f(0) > 0$ . Moreover, we note that  $f(s)$  satisfies  $(C_1)$ ,  $(C_2)$  and  $\lim_{s \rightarrow 1^-} \frac{f(s)}{1-s} = 0$ . The following is similar to Example 1, so we omit it.

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## Minkowski 空间中一维给定平均曲率方程正解的 $\Sigma$ -型连通分支

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**摘要:** 运用上下解方法证明 Minkowski 空间中一维给定平均曲率方程 
$$\begin{cases} -\left(\frac{u'(t)}{\sqrt{1-(u'(t))^2}}\right)' = \lambda h(t)f(u(t)), & t \in (0, 1), \\ u(0) = 0, & u'(1) + \sqrt{\lambda} u(1) = 0, \end{cases}$$

在  $f(0)=0$  和  $f(0)>0$  两种情形下正解集  $\Sigma$  型连通分支的存在性, 其中,  $\lambda>0$  为参数,  $f \in C^2([0, \infty), \mathbb{R})$  单调递增且满足  $\lim_{u \rightarrow 1^-} \frac{f(u)}{1-u} = 0$ ,  $h \in C^1([0, 1], (0, \infty))$  是单调递减函数且  $h(t)>1$ .

**关键词:** 边界条件带参数; 正解; 上下解定理; 渐进性质

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