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Study on the Density-Independent Fractional Diffusion-Reaction Equation with the Beta Derivative

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Abstract: In this paper, the density-independent fractional diffusion-reaction (FDR) equation involving quadratic nonlinearity is investigated. The fractional derivative is illustrated in the beta derivative sense. We firstly propose Bernoulli (G'/G)-expansion method to study nonlinear fractional differential equations (NFDEs). Subsequently, closed form solutions of the density-independent FDR equation are acquired successfully. In order to better understand the dynamic behaviors of these solutions, 3D, contour map and line plots are given by the computer simulation. The results show that the proposed method is a reliable and efficient approach.

Key words: density-independent fractional diffusion-reaction (FDR) equation; beta derivative; closed form solutions; Bernoulli (G'/G)-expansion method

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0 Introduction

In the real world, because of the complexity of things, linear systems are often only theoretical approximation of some simple nonlinear systems. However, nonlinear systems capture the fundamental nature of the objective world more accurately. Therefore, it is of great significance to understand and study nonlinear phenomena for the development of modern science and technology. In recent years, nonlinear differential equations have been extensively studied and widely applied to numerous domains within the natural sciences, establishing itself as a central focus of modern scientific research. From the perspective of mathematical physics, many nonlinear phenomena can be reduced to solving nonlinear differential equations[1-2]. Among of them, nonlinear fractional differential equations (NFDEs) have attracted extensive attention. NFDEs are widely used in fluid mechanics, turbulence and viscoelasticity, anomalous diffusion, fractal and dispersion in porous media, signal processing and system identification, electromagnetic waves and other fields. With the further application of NFDEs, finding their closed form solutions is still the primary goal. In the literatures, a lot of effective techniques have been constructed to search closed form solutions of NFDEs such as F-expansion method[3], fractional sub-equation method[4-6], first integral method[7-8], Kudryashov methods[9-10], $\exp(-\varphi(z))$-expansion method[11-12], (G'/G)-expansion method[13-16], tanh-function method[17-18], truncated Painlevé expansion

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method[19], Sine-Gordon expansion method[20], generalized Riccati equation mapping method[21], modified simple equation method[22-24], and complex method[25-27], multivariate bilinear neural network method[28] and so on.

Fractional derivatives have various definitions, such as the beta derivative, conformable derivative, and Riemann-Liouville derivative. These different types of fractional derivatives offer unique perspectives and computational approaches. The diversity of fractional derivatives has drawn significant interest from researchers in studying and solving this class of equations. Uddin *et al*[29] derived exact solutions for the fractional generalized Duffing model. Hosseini *et al*[30] investigated the density-dependent conformable fractional diffusion-reaction equation, and employed two distinct methods to obtain exact solutions. Rezazadeh *et al*[31] addressed the same equation using the first integral method. Additionally, Sene and Fall[32], through the application of the Laplace transform method, provided approximate solutions for fractional diffusion equations.

Wang *et al*[33] proposed (G'/G)-expansion method for studying nonlinear differential equations. Bernoulli (G'/G)-expansion method, inspired by this approach, assumed that the traveling wave solutions of a nonlinear evolution equation can be expressed as polynomials of (G'/G), where G satisfies the Bernoulli differential equation. We employ Bernoulli (G'/G)-expansion method to analyze the density-independent FDR equation with the beta derivative for quadratic nonlinearity, as described in Ref.[34] and generalized into fractional derivative in Ref.[35]. The equation is given by

$$D_t^{\beta}u+\eta D_x^{\beta}u=\zeta D_x^{2\beta}u+\alpha u-\lambda u^2, 0<\beta\leqslant 1, \tag{1}$$

where $u = u(x,\ t)$ is the concentration or density variable, depending on the phenomenon under study, ζ is the diffusion coefficients, D is the fractional differential operator, and $\eta,\ \alpha,\ \lambda$ are real constants.

Eq. (1) is of interest in the field of population biology[36]. Additionally, it can be considered as a generalization of the Fisher equation[37]. Kumar *et al*[34] exploited Eq. (1) with auxiliary equation method in integer differential derivative. Kumar *et al*[35] discussed Eq. (1) using the conformable fractional derivative along with the modified Kudryashov method. However, the properties of this equation under the beta derivative remain insufficiently studied. In this work, we analyze the same model using the beta fractional derivative to further explore the local properties of the FDR system.

This paper is organized as follows. Section 1 presents the detailed steps for transformation and the Bernoulli (G'/G)-expansion method; Section 2 describes the applications to the density-independent FDR equation; Section 3 conveys a concise conclusion.

1 Proposal of the Bernoulli (G'/G)-Expansion Method

At first, some properties of the beta fractional derivative are introduced and the Bernoulli (G'/G)-expansion method is explained in detail in order to better understand the results.

Fractional derivatives, as global operators in generalized fractional calculus, combine differentiation with convolution integrals to capture non-locality and memory effects. In contrast, the beta derivative, derived from traditional integer-order derivatives with an added nonlinear term, lacks the characteristics of a global operator. While fractional derivatives inherently describe long-range dependencies and historical effects, the beta derivative is more localized, making it suitable for scenarios where global properties are less relevant.

Let $f(z)$ is specified as a function of all non-negative z with the β derivative[38-39], hence

$$D_z^{\beta}\left(f\left(z\right)\right)=\frac{\mathrm{d}^{\beta}f\left(z\right)}{\mathrm{d}z^{\beta}}=\lim_{\delta\to 0}\frac{f(z+\delta(z+\dfrac{1}{\Gamma\left(\beta\right)})^{1-\beta})-f(z)}{\delta},$$

$0<\beta\leqslant 1$, in which $\Gamma(\beta)$ is the Gamma function.

Some effective properties of above definition[38-41] are included

$$D_z^{\beta}(f(z))=(z+\frac{1}{\Gamma\left(\beta\right)})^{1-\beta}\frac{\mathrm{d}f\left(z\right)}{\mathrm{d}z},$$

$$D_z^{\beta}(f\circ g(z))=(z+\frac{1}{\Gamma\left(\beta\right)})^{1-\beta}g'(z)f'(g(z)).$$

By using the knowledge of fractional derivative and the Bernoulli (G'/G)-expansion method, the closed-form solutions of FDR equation can be obtianed conveniently. Seeking closed form solutions of FDR equation is facilitated by exerting the Bernoulli (G'/G)-expansion method which is presented according to the following steps.

Consider the space-time NFDE as follows

$$F\left(u,D_x^{\beta}u,D_t^{\beta}u,D_x^{\beta}D_t^{\beta}u,D_x^{2\beta}u,\cdots\right)=0, 0<\beta<1, \tag{2}$$

in which F is the polynomial of unknown function u and its fractional derivatives.

Step 1 Insert the transformation

$$u\left(x,t\right)=w\left(\xi\right),\xi=\frac{k}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta}-\frac{c}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta} \tag{3}$$

into space-time NFDE (2) to yield

$$P\left(w,w',w'',\cdots\right)=0, \tag{4}$$

where $k,\ c$ are constants that may relate to wave speed, and P is the polynomial of w along with its derivatives.

Step 2 Assume that (4) has the following solution

$$w=a_0+\sum_{i=1}^{n}a_i\left(\frac{G'}{G}\right)^i,\ a_n\neq 0, \tag{5}$$

where a_i for $i=0,\ 1,\cdots,\ n$ are constants.

To describe Bernoulli (G'/G)-expansion method, the following Bernoulli equation for $G=G(\xi)$ needs to be introduced:

$$G'+pG=qG^2, \tag{6}$$

where p and q are constants $(q\neq 0)$ can be specified later.

Step 3 In Eq. (4), by balancing the highest order derivative of w and highest order nonlinear term, we obtian integer n. By replacing (5) and (6) into (4), a system of algebra equations will be acquired through the gathering of G with the same order.

Step 4 Inserting the results of above steps into (5) and using the following general solutions of (6):

$$\left(\frac{G'}{G}\right)=-\frac{pA_1\exp\left(p\xi\right)}{A_1\exp\left(p\xi\right)+q}, \tag{7}$$

provided A_1 is an integral constant, we achieve closed form solutions of Eq. (2).

2 Applications to the Density-Independent FDR Equation

By inserting transformation (3) in Eq. (1), we can derive an ordinary differential equation as below

$$k^2\zeta w''+\left(c-k\eta\right)w'+\alpha w-\lambda w^2=0. \tag{8}$$

By balancing between w'' and w^2 in Eq. (8), $n=2$ can be determined for (5), which yields the solution of w as follows:

$$w=a_0+a_1\left(\frac{G'}{G}\right)+a_2\left(\frac{G'}{G}\right)^2, \tag{9}$$

where $G=G(\xi)$ is satisfied with (6). To gain the meaningful solutions, displacing (9) along with (6) into (8), with collecting coefficients of the same order of G_j $(j=0,\cdots,4)$ to zero, a system of algebraic equations is attained as follows:

$$6a_2k^2q^4\zeta-a_2^2q^4\lambda=0,$$

$$\begin{aligned}&2a_1a_2q^3\lambda+2a_1k^2q^3\zeta+4a_2^2pq^3\lambda+2a_2cq^3-14a_2k^2pq^3\zeta\\&-2a_2kq^3\eta=0,\end{aligned}$$

$$\begin{aligned}&6a_1a_2pq^2\lambda-2a_0a_2q^2\lambda-a_1^2q^2\lambda+a_1cq^2-3a_1k^2pq^2\zeta\\&-a_1kq^2\eta-6a_2^2p^2q^2\lambda-4a_2cpq^2+10a_2k^2p^2q^2\zeta\\&+4a_2kpq^2\eta+\alpha a_2q^2=0,\end{aligned}$$

$$\begin{aligned}&4a_0a_2pq\lambda-2a_0a_1q\lambda+2a_1^2pq\lambda-6a_1a_2p^2q\lambda-a_1cpq\\&+a_1k^2p^2q\zeta+a_1kpq\eta+\alpha a_1q+4a_2^2p^3q\lambda+2a_2cp^2q\\&-2a_2k^2p^3q\zeta-2a_2kp^2q\eta-2\alpha a_2pq=0,\end{aligned}$$

$$\begin{aligned}&2a_0a_1p\lambda-a_0^2\lambda-2a_0a_2p^2\lambda+\alpha a_0-a_1^2p^2\lambda+2a_1a_2p^3\lambda\\&-\alpha a_1p-a_2^2p^4\lambda+\alpha a_2p^2=0.\end{aligned}$$

The begotten results for coefficients are as below.

Family 1: $a_0=\dfrac{\alpha}{\lambda},\ a_1=\pm\dfrac{2\sqrt{6}\sqrt{\alpha\zeta}\,k}{\lambda},\ a_2=\dfrac{6\zeta k^2}{\lambda},$

$$c=\pm\frac{5\sqrt{6}\sqrt{\alpha\zeta}\,k}{6}+\eta k,\ p=\pm\frac{\sqrt{6}\sqrt{\alpha\zeta}}{6\zeta k}.$$

Replacing these results into (9) and by using the definition of (G'/G), the following solutions are extracted for (1):

$$\begin{aligned}w_{1,1}=&\frac{6\zeta k^2}{\lambda}\left(\frac{pA_1\exp\left(p\xi\right)}{A_1\exp\left(p\xi\right)+q}\right)^2\\&-\frac{2\sqrt{6}\sqrt{\alpha\zeta}\,k}{\lambda}\left(\frac{pA_1\exp\left(p\xi\right)}{A_1\exp\left(p\xi\right)+q}\right)+\frac{\alpha}{\lambda},\end{aligned} \tag{10}$$

$$\begin{aligned}w_{1,2}=&\frac{6\zeta k^2}{\lambda}\left(\frac{pA_1\exp\left(p\xi\right)}{A_1\exp\left(p\xi\right)+q}\right)^2\\&+\frac{2\sqrt{6}\sqrt{\alpha\zeta}\,k}{\lambda}\left(\frac{pA_1\exp\left(p\xi\right)}{A_1\exp\left(p\xi\right)+q}\right)+\frac{\alpha}{\lambda},\end{aligned} \tag{11}$$

where A_1 and q are arbitrary constants and $p=\dfrac{\sqrt{6}\sqrt{\alpha\zeta}}{6\zeta k},p=-\dfrac{\sqrt{6}\sqrt{\alpha\zeta}}{6\zeta k}$, respectively.

Dynamic behaviors of $w_{1,1}$ and $w_{1,2}$ are displayed in Figs. 1 and 2, by considering $A_1=1$, $k=2$, $q=0.3$, $\lambda=3$, $\eta=0.5$, $\zeta=0.4$ and $\alpha=0.2$, for different values of $\beta=0.1$, $\beta=0.4$ and $\beta=0.7$.

Family 2: $a_0=0,\ a_1=\pm\dfrac{2k\sqrt{6}\sqrt{-\alpha\zeta}}{\lambda},\ a_2=\dfrac{6\zeta k^2}{\lambda},$

$$c=\pm\frac{5\sqrt{-6\alpha\zeta}\,k}{6}+\eta k,\ p=\pm\frac{\sqrt{6}\sqrt{-\alpha\zeta}}{6\zeta k}.$$

Replacing these results into (9) and by using the definition of (G'/G), the following solutions are extracted for (1):

$$\begin{aligned}w_{2,1}=&\frac{6\zeta k^2}{\lambda}\left(\frac{pA_1\exp\left(p\xi\right)}{A_1\exp\left(p\xi\right)+q}\right)^2\\&-\frac{2k\sqrt{6}\sqrt{-\alpha\zeta}}{\lambda}\left(\frac{pA_1\exp\left(p\xi\right)}{A_1\exp\left(p\xi\right)+q}\right),\end{aligned} \tag{12}$$

$$\begin{aligned}w_{2,2}=&\frac{6\zeta k^2}{\lambda}\left(\frac{pA_1\exp\left(p\xi\right)}{A_1\exp\left(p\xi\right)+q}\right)^2\\&+\frac{2k\sqrt{6}\sqrt{-\alpha\zeta}}{\lambda}\left(\frac{pA_1\exp\left(p\xi\right)}{A_1\exp\left(p\xi\right)+q}\right),\end{aligned} \tag{13}$$

where A_1 and q are constants and $p=\dfrac{\sqrt{6}\sqrt{-\alpha\zeta}}{6\zeta k},p=-\dfrac{\sqrt{6}\sqrt{-\alpha\zeta}}{6\zeta k}$, respectively.

Figures 3 and 4 are displayed for exact solution of $w_{2,1}$ and $w_{2,2}$, respectively, in 3D, contour map and line plots by setting $A_1=1$, $k=2$, $\alpha=-0.2$, $\lambda=3$, $\zeta=0.4$, $\eta=0.5$, $q=0.3$, for different values of $\beta=0.1$, $\beta=0.4$ and $\beta=0.7$.

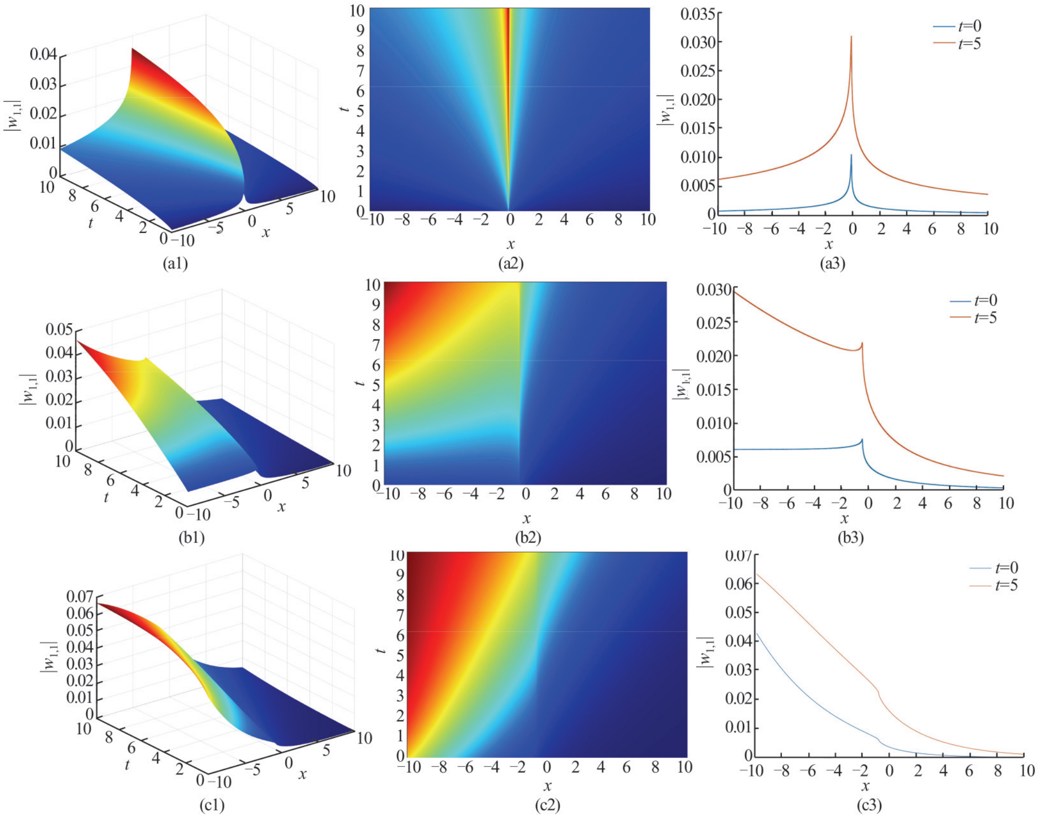

Fig. 1 Closed form solution $w_{1,1}$ of Eq. (1), where panels (a1-a3) correspond to $\beta = 0.1$, (b1-b3) to $\beta = 0.4$ and (c1-c3) to $\beta = 0.7$

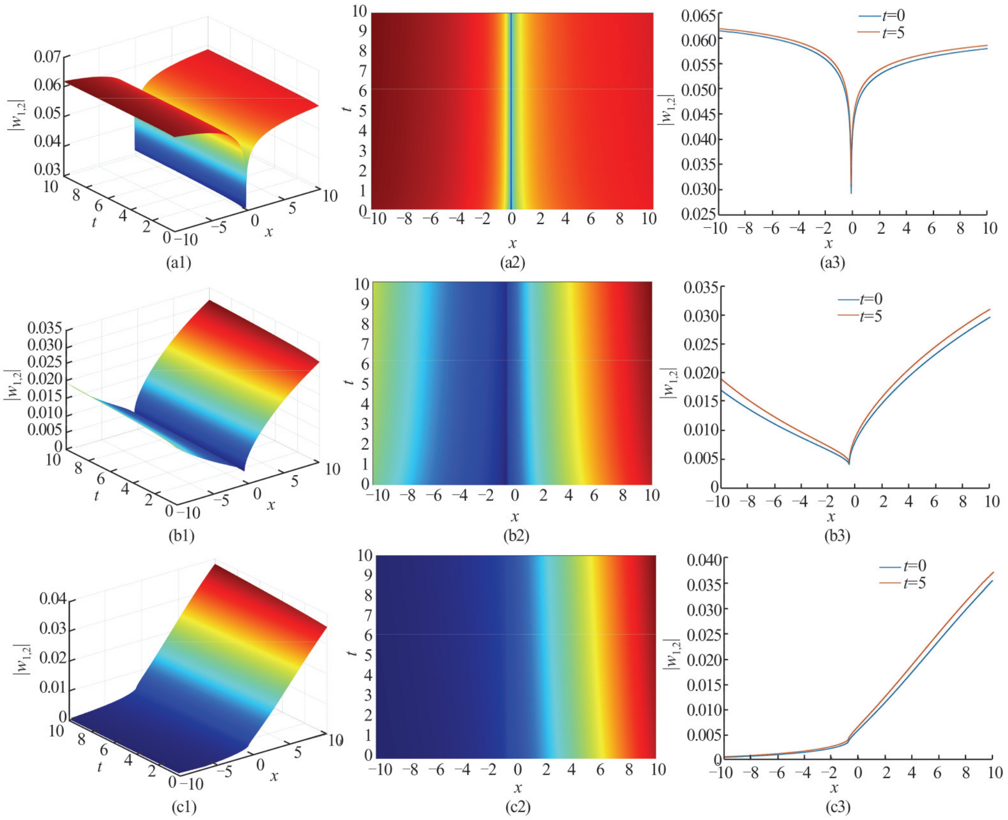

Fig. 2 Closed form solution $w_{1,2}$ of Eq. (1), where panels (a1-a3) correspond to $\beta = 0.1$, (b1-b3) to $\beta = 0.4$ and (c1-c3) to $\beta = 0.7$

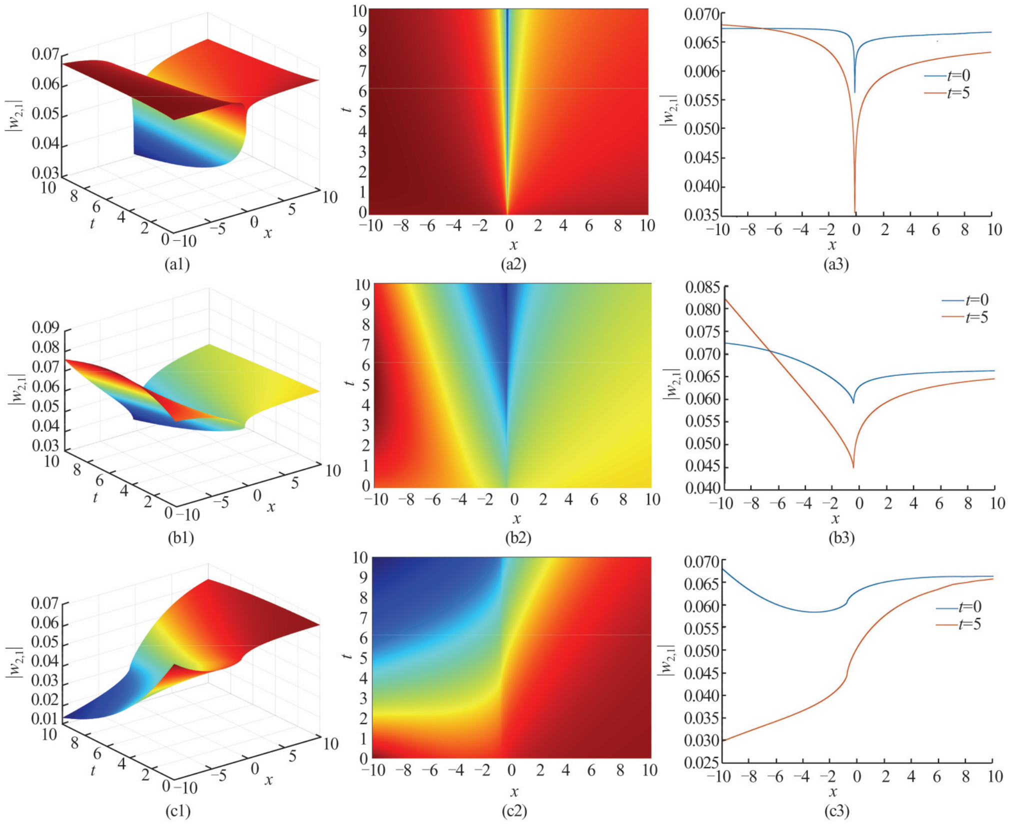

Fig. 3 Closed form solution $w_{2,1}$ of Eq. (1), where panels (a1-a3) correspond to β = 0.1, (b1-b3) to β = 0.4 and (c1-c3) to β = 0.7

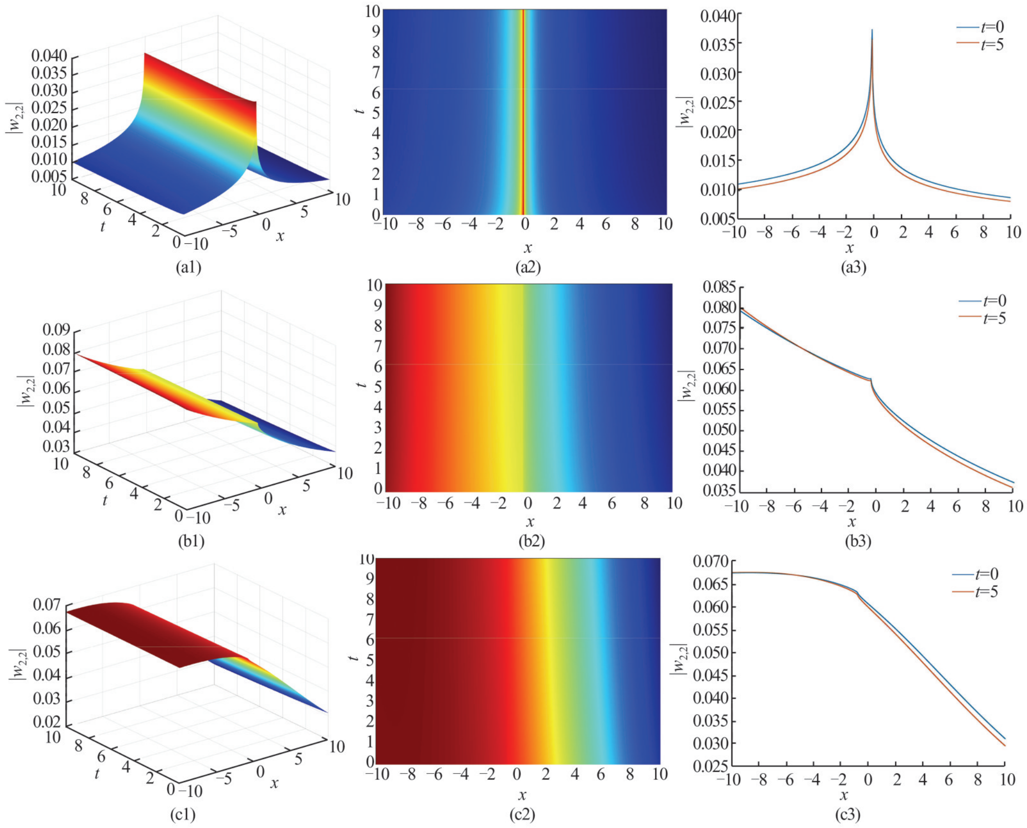

Fig. 4 Closed form solution $w_{2,2}$ of Eq. (1), where panels (a1-a3) correspond to β = 0.1, (b1-b3) to β = 0.4 and (c1-c3) to β = 0.7

Family 3: $a_0=0,\ a_1=0,\ a_2=\dfrac{6\zeta k^2}{\lambda},$

$$c=\pm\frac{5\sqrt{6}\sqrt{\alpha\zeta}\,k}{6}+\eta k,\ p=\mp\frac{\sqrt{6}\sqrt{\alpha\zeta}}{6\zeta k}.$$

Replacing these results into (9) and by using the definition of (G'/G), the following solutions are extracted for Eq. (1),

$$w_{3,1}=\frac{6\zeta k^2}{\lambda}\left(\frac{pA_1\exp\left(p\xi\right)}{A_1\exp\left(p\xi\right)+q}\right)^2, \tag{14}$$

$$w_{3,2}=\frac{6\zeta k^2}{\lambda}\left(\frac{pA_1\exp\left(p\xi\right)}{A_1\exp\left(p\xi\right)+q}\right)^2, \tag{15}$$

where A_1 and q are arbitrary constants and $p=-\dfrac{\sqrt{6}\sqrt{\alpha\zeta}}{6\zeta k}, p=\dfrac{\sqrt{6}\sqrt{\alpha\zeta}}{6\zeta k}$, respectively.

Figures 5 and 6 conclude 3D, contour map and line plots of $w_{3,1}$ and $w_{3,2}$, respectively, by setting $A_1=1.1$, $k=2$, $\alpha=0.01$, $\lambda=0.4$, $\eta=0.2$, $\zeta=1$, $q=0.5$ for different values of $\beta=0.1$, $\beta=0.4$ and $\beta=0.7$.

Family 4: $a_0=\dfrac{\alpha}{\lambda},\ a_1=0,\ a_2=\dfrac{6\zeta k^2}{\lambda},$

$$c=\pm\frac{5\sqrt{-6\alpha\zeta}\,k}{6}+\eta k,\ p=\mp\frac{\sqrt{6}\sqrt{-\alpha\zeta}}{6\zeta k}.$$

Replacing these results into (9) and by using the definition of (G'/G), the following solutions are extracted for Eq. (1),

$$w_{4,1}=\frac{6\zeta k^2}{\lambda}\left(\frac{pA_1\exp\left(p\xi\right)}{A_1\exp\left(p\xi\right)+q}\right)^2+\frac{\alpha}{\lambda}, \tag{16}$$

$$w_{4,2}=\frac{6\zeta k^2}{\lambda}\left(\frac{pA_1\exp\left(p\xi\right)}{A_1\exp\left(p\xi\right)+q}\right)^2+\frac{\alpha}{\lambda}, \tag{17}$$

where A_1 and q are constants and $p=-\dfrac{\sqrt{6}\sqrt{-\alpha\zeta}}{6\zeta k}, p=\dfrac{\sqrt{6}\sqrt{-\alpha\zeta}}{6\zeta k}$, respectively.

We show 3D, contour map and line plots of $w_{4,1}$ and $w_{4,2}$ in Fig. 7 and Fig. 8, respectively, by exerting $A_1=1.3$, $k=2$, $\alpha=-0.01$, $\lambda=1.4$, $\eta=2.2$, $\zeta=0.1$, $q=0.5$ for different values of (a1-a3) $\beta=0.1$, (b1-b3) $\beta=0.4$ and (c1-c3) $\beta=0.7$.

We find that the parameter β mainly affects the negative part of x in Figs. 1, 2, 6, 7 and 8, while in Figs. 3, 4, 5, β influences the entire region. As shown in the figures, an increase in β results in a decrease of the solutions in the negative half-plane of x. Moreover, for two solutions within a given family, the trends in the

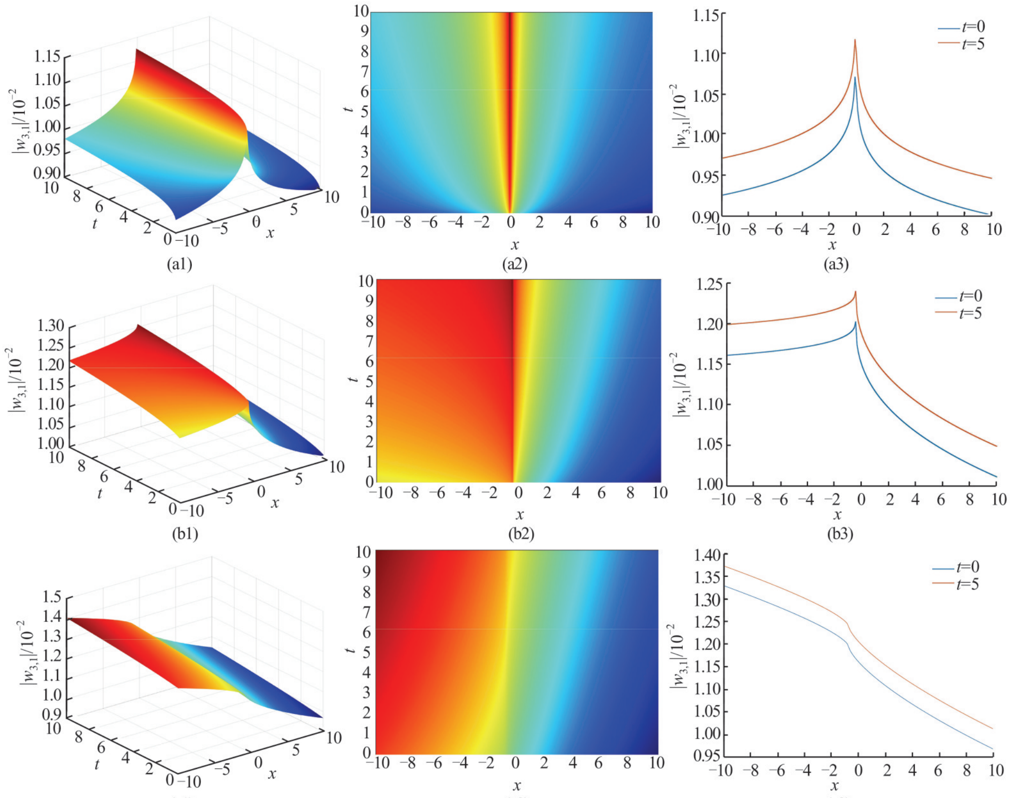

Fig. 5 Closed form solution $w_{3,1}$ of Eq. (1), where panels (a1-a3) correspond to β = 0.1, (b1-b3) to β = 0.4 and (c1-c3) to β = 0.7

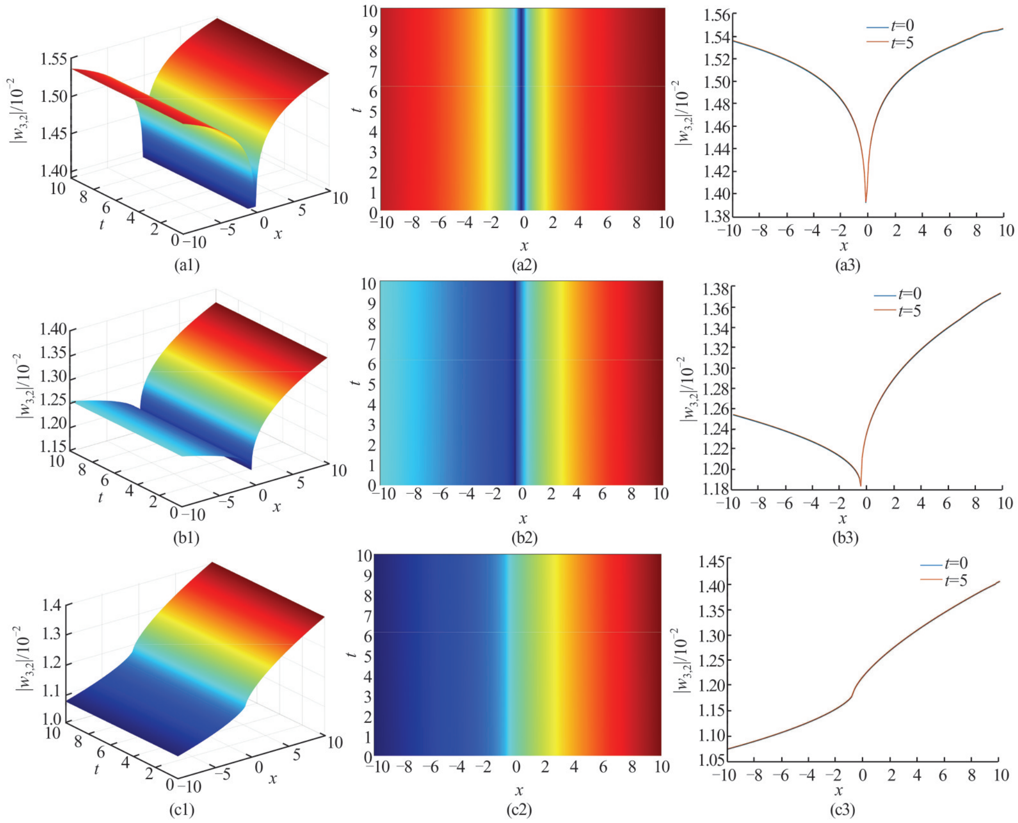

Fig. 6 Closed form solution $w_{3,2}$ of Eq. (1), where panels (a1-a3) correspond to $\beta = 0.1$, (b1-b3) to $\beta = 0.4$ and (c1-c3) to $\beta = 0.7$

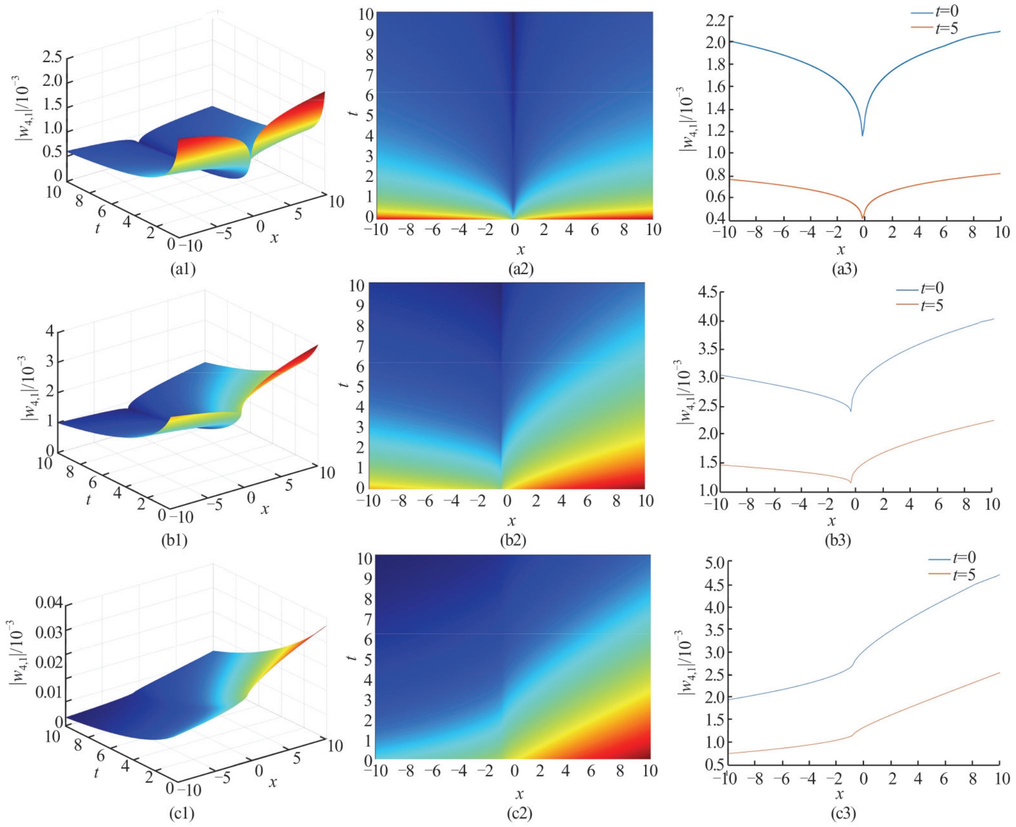

Fig. 7 Closed form solution $w_{4,1}$ of Eq. (1), where panels (a1-a3) correspond to $\beta = 0.1$, (b1-b3) to $\beta = 0.4$ and (c1-c3) to $\beta = 0.7$

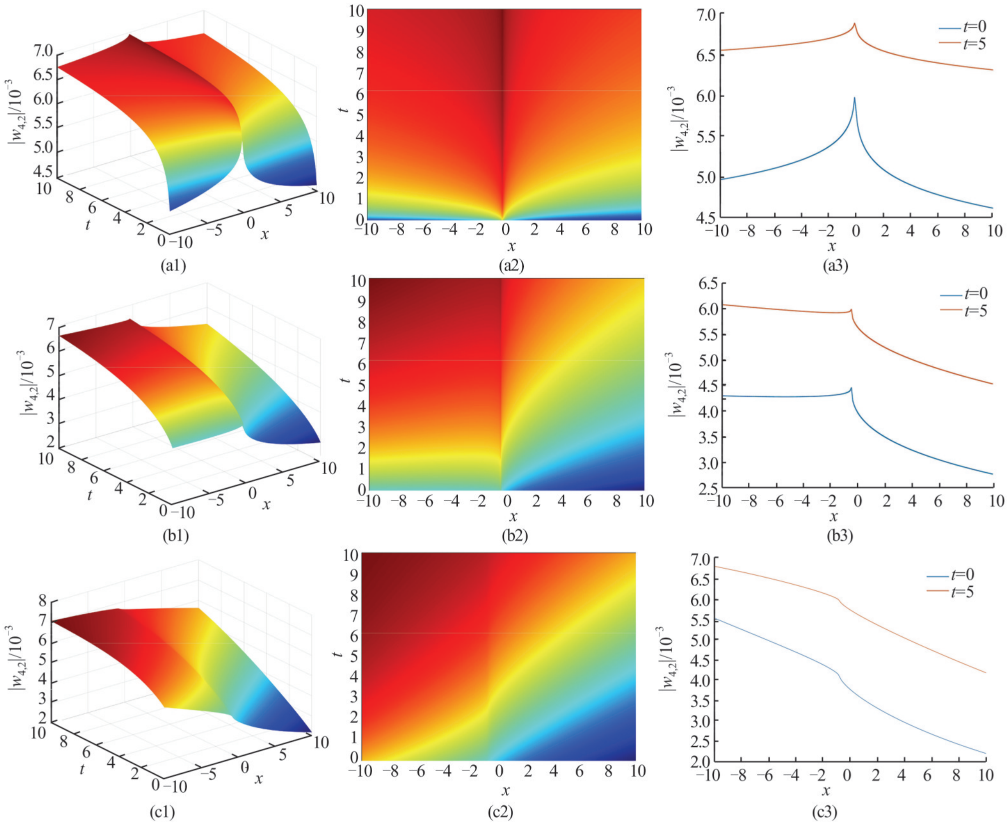

Fig. 8 Closed form solution $w_{4,2}$ of Eq. (1), where panels (a1-a3) correspond to $\beta = 0.1$, (b1-b3) to $\beta = 0.4$ and (c1-c3) to $\beta = 0.7$

changes of convexity and concavity oppose each other as β increases. These observations may be related to the inherent properties of Eq. (1). Additionally, the simulations reveal that the solutions $w_{1,1}$, $w_{2,2}$, $w_{3,1}$, $w_{4,2}$ exhibit bright soliton dynamic in certain cases, while $w_{1,2}$, $w_{2,1}$, $w_{3,2}$, $w_{4,1}$ show dark soliton dynamic. The different behaviors of the solution may be useful to the population prediction, showing the peek and valley of the population.

3 Conclusion

In this article, due to the use of wave transformation which contains $\Gamma(\beta)$ function, the complexity of FDE could be reduced to ordinary differential equation. We find eight new rational exponent solutions to Eq. (1) with Bernoulli (G'/G)-expansion method. We derived bright and dark soliton solutions of density-independent FDR equation with quadratic nonlinearity by use of the Bernoulli (G'/G)-expansion method. Moreover, 3D, contour map and line plots were demonstrated to show the effect of different values of the derivative of order β on wave structures. The influences of parameter β is extracted by comparing the simulations figures. Parameter β largely affects the negative region of x. Additionally, β influences the trends in the convexity and concavity. By these results, we predict that the Bernoulli (G'/G)-expansion method can be satisfied enormous range of fractional differential equations.

Although we have obtained numerous results regarding the solutions, the real-world applications of this model still require further investigation.

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带有beta导数的密度无关分数阶扩散反应方程研究

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摘要：本文研究了涉及二次非线性的密度无关分数阶扩散反应方程。分数阶导数以beta导数的形式表示。首先，提出了Bernoulli (G'/G)-展开法，并用其研究非线性分数阶微分方程。然后，获得了密度无关方程的精确解。为了更好地了解这些解的动力学行为，通过计算机仿真给出了三维图、等高线图和线图。结果表明，所提出的方法是一种可靠且高效的研究方法。

关键词：密度无关分数阶扩散反应方程；beta导数；精确解；Bernoulli (G'/G)-展开法

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