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Spectral Properties of Dirac Operator with λ-Dependent Boundary Condition

□ **ZHONG Linlu, DU Gaofeng†**

College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, Gansu, China

Abstract: In this study, we mainly discuss some spectral properties of the Dirac operator with eigenparameter-dependent boundary condition. Initially, we reformulate the spectral problem into linear operator eigenparameter problem in a suitable Hilbert space, and obtain some pivotal properties of self-adjoint operator. Subsequently, by establishing the boundary condition space and constructing the embedded mapping, we show that the simple eigenvalue branch of this system is not only continuous, but also smooth. We then obtain the differential expressions of the eigenvalue branch in the sense of Fréchet derivative.

Key words: λ-dependent boundary condition; spectral properties; Dirac operator

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0 Introduction

Dirac operators are important models in quantum mechanics. As a result, some conclusions about these operators have been obtained (see Refs. [1-4]). As a typical problem in the spectral theory of differential operators, Dirac operators are closely related to Sturm-Liouville operators, and they have many similarities in the properties and research methods of eigenvalues, see Refs. [5-9]. In particular, in Ref. [10], Li *et al* studied the continuous dependence of eigenvalue of self-adjoint Dirac system consisting of the symmetric differential operator

$$\boldsymbol{B}\mathcal{U}'(x)+\boldsymbol{Q}(x)\mathcal{U}(x)=\lambda\mathcal{U}(x),$$

$$x\in J\equiv(\alpha',\beta'),\ -\infty\leqslant\alpha'<\beta'\leqslant+\infty, \quad (1)$$

$$\boldsymbol{M}\mathcal{U}(\alpha)+\boldsymbol{N}\mathcal{U}(\beta)=0,\ \mathcal{U}(x)=(u_1(x),u_2(x))^{\mathrm{T}}, \quad (2)$$

where $\boldsymbol{B}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, $\boldsymbol{Q}(x)=\begin{pmatrix}\rho(x)&\sigma(x)\\\sigma(x)&\varrho(x)\end{pmatrix}\in L_1(J)$ and $\boldsymbol{Q}(x)$ is real-valued Lebesgue measurable function on J; $\boldsymbol{M}$ and $\boldsymbol{N}$ are 2×2 complex matrices such that $\operatorname{rank}(\boldsymbol{M},\boldsymbol{N})=2$ and satisfy $\boldsymbol{MEM}^*=\boldsymbol{NEN}^*$, where $\boldsymbol{M}^*$ denotes the complex conjugate transpose of $\boldsymbol{M}$, $\boldsymbol{E}$ is the second order symplectic matrix. Besides, we are also interested in the spectral analysis of Dirac operator with the spectral parameter in boundary condition, which has a discrete spectrum consisting of an increasing infinite sequence of (real, simple) eigenvalues $\lambda_{\pm n}$ such that $\lambda_n\to\pm\infty$ as $n\to\infty$, see Ref. [11], and some remarkable works have been reached, see Refs. [12-13].

From the above literature, we notice that the research on problems (1)-(2) requires that the boundary conditions satisfy $\boldsymbol{MEM}^*=\boldsymbol{NEN}^*$. Typically, the eigenparameter is only present in differential equations. None-

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Biography: ZHONG Linlu, female, Master candidate, research direction: ordinary differential equations and dynamic systems. E-mail: rqzhong0510@163.com

† Corresponding author. E-mail: dgf96520@163.com

theless, in numerous practical applications, including mechanics and acoustic scattering theory, it is necessary for the spectral parameter to be featured not only within the differential equations but also within the boundary conditions, see Ref. [14]. Recently, inverse spectral problems for Sturm-Liouville operators with non-self-adjoint eigenparameter-dependent boundary conditions have been studied via matrix representations and inverse matrix eigenvalue problems, see Ref. [15]. For the above model the condition $\boldsymbol{MEM}^*=\boldsymbol{NEN}^*$ may not be satisfied. Therefore, we hope to get the correlation spectrum properties of this kind of problems. For example, considering the self-adjointness of operators and the continuous dependence of eigenvalues, it is necessary to discuss the eigenvalue problem of a linear operator in a suitable Hilbert space. To address the limitation that these boundary conditions lack self-adjointness, we construct a suitable Banach space and an embedded mapping, and prove the continuity of the embedded mapping. Moreover, based on the definition of Fréchet derivative, we obtain the differentiability of eigenvalue bifurcation.

Our research plan is structured as follows. In Section 1, we introduce a self-adjoint operator eigenvalue problem and elucidate its critical spectral properties from various perspectives. In Section 2 and 3, leveraging the established continuity of the embedded mapping, we demonstrate the differentiability of the eigenvalue branch and present the corresponding derivative formulas.

1 Some Properties

This paper concerns eigenvalue problem of the form

$$\mathcal{LU}:=\boldsymbol{B}\mathcal{U}'(x)+\boldsymbol{Q}(x)\mathcal{U}(x)=\lambda\mathcal{U}(x),\quad -\infty<a<b<+\infty, \tag{3}$$

$$\boldsymbol{M}\mathcal{U}(a)+\boldsymbol{N}\mathcal{U}(b)=0,\ \mathcal{U}(x)=\left(u_1(x),u_2(x)\right)^{\mathrm{T}}, \tag{4}$$

where $\boldsymbol{Q}(x)=[p(x), q(x); q(x), p(x)]$ and every element in $\boldsymbol{Q}(x)$ belongs to $L^1_{\mathrm{loc}}(a,b)$, λ is the spectral parameter. Here, we require $\boldsymbol{M}$, $\boldsymbol{N}$ to satisfy $\boldsymbol{M}=[a_0+a_1\lambda, -(b_0+b_1\lambda); 0,0]$, $\boldsymbol{N}=[0,0; c_o+c_1\lambda, -(d_0+d_1\lambda)]$ and a_i, b_i, c_i, d_i, $i=0,1$, are real numbers satisfying $\sigma_0=a_0b_1-b_0a_1>0$ and $\sigma_1=c_0d_1-d_0c_1<0$.

We first consider a linear operator eigenvalue problem derived from spectral problems (3)-(4). The inner product in the Hilbert space $H=L_2(a,b)\oplus L_2(a,b)\oplus\mathbb{C}^2$ associated with

$$\langle \boldsymbol{U},\boldsymbol{Z}\rangle=\int_a^b(u_1(x)\bar{z}_1(x)+u_2(x)\bar{z}_2(x))\,\mathrm{d}x-\frac{1}{\sigma_0}u_3\bar{z}_3+\frac{1}{\sigma_1}u_4\bar{z}_4$$

for $\boldsymbol{U}=(u_1(x),u_2(x),u_3,u_4)^{\mathrm{T}}$, $\boldsymbol{Z}=(z_1(x), z_2(x),z_3,z_4)^{\mathrm{T}}\in H$. Define the operator $\mathcal{L}$ acting in H so that

$$\mathcal{L}\boldsymbol{U}=\begin{pmatrix}\boldsymbol{B}\mathcal{U}'(x)+\boldsymbol{Q}(x)\mathcal{U}(x)\\ b_0u_2(a)-a_0u_1(a)\\ d_0u_2(b)-c_0u_1(b)\end{pmatrix}$$

with the domain

$$D(\mathcal{L})=\{\boldsymbol{U}\in H:\ u_1,u_2\in \mathrm{AC}[a,b] \text{ and satisfy boundary condition (4)},\ \mathcal{LU}\in L_2(a,b)\oplus L_2(a,b),\ u_3=a_1u_1(a)-b_1u_2(a)\in\mathbb{C},\ u_4=c_1u_1(b)-d_1u_2(b)\in\mathbb{C}\}.$$

Here $\mathrm{AC}[a,b]$ denotes the set of absolutely continuous and complex-valued functions on $[a,b]$.

By immediate verification, we conclude that the problems (3)-(4) are equivalent to linear operator eigenvalue problem $\mathcal{L}\boldsymbol{U}=\lambda\boldsymbol{U}$. We now focus on discussing the properties of linear operator $\mathcal{L}$ as follows.

Theorem 1 $\mathcal{L}$ is a self-adjoint operator in H.

Proof In order to prove that the operator $\mathcal{L}$ is self-adjoint, we first need to prove that the domain of operator $\mathcal{L}$ is dense in H. Suppose $\boldsymbol{U}=(u_1(x),u_2(x),u_3,u_4)^{\mathrm{T}}\in H$ and $\boldsymbol{U}$ is orthogonal to $\boldsymbol{Y}=(y_1(x), y_2(x), y_3, y_4)^{\mathrm{T}}\in D(\mathcal{L})$. We will prove $\boldsymbol{U}=(0,0,0,0)^{\mathrm{T}}$. Since $C_0^\infty\oplus C_0^\infty\oplus\{0\}\oplus\{0\}\subset D(\mathcal{L})$, for arbitrary $\boldsymbol{V}=(v_1(x),v_2(x),0,0)^{\mathrm{T}}\in C_0^\infty\oplus C_0^\infty\oplus\{0\}\oplus\{0\}$, we have $\langle \boldsymbol{U},\boldsymbol{V}\rangle=\int_a^b(u_1(x)\bar{v}_1(x)+u_2(x)\bar{v}_2(x))\,\mathrm{d}x=0$. Since C_0^∞ is dense in $L_2(a,b)$, we have $u_1(x)=u_2(x)=0$, so $\boldsymbol{U}=(0, 0, u_3, u_4)^{\mathrm{T}}$. For any $\boldsymbol{Y}$, through the inner product in H, we get $\langle \boldsymbol{U},\boldsymbol{Y}\rangle=\frac{1}{\sigma_1}u_4\bar{y}_4-\frac{1}{\sigma_0}u_3\bar{y}_3=0$. Since $\boldsymbol{Y}$ is arbitrary, we have $u_3=u_4=0$. Hence $\boldsymbol{U}=(0,0,0,0)^{\mathrm{T}}$.

Secondly, we need to show that the operator $\mathcal{L}$ is symmetric. Let $\boldsymbol{U}=(u_1(x), u_2(x), u_3, u_4)^{\mathrm{T}}$. For any $\boldsymbol{V}=(v_1(x),v_2(x),v_3,v_4)^{\mathrm{T}}\in D(\mathcal{L})$, then

$$\begin{aligned}\langle\mathcal{L}\boldsymbol{U},\boldsymbol{V}\rangle-\langle\boldsymbol{U},\mathcal{L}\boldsymbol{V}\rangle&=\bar{v}_1(b)u_2(b)-\bar{v}_2(b)u_1(b)\\&+u_1(a)\bar{v}_2(a)-u_2(a)\bar{v}_1(a)-\frac{1}{\sigma_0}(b_0u_2(a)-a_0u_1(a))\bar{v}_3\\&+\frac{1}{\sigma_1}(d_0u_2(b)-c_0u_1(b))\bar{v}_4+\frac{1}{\sigma_0}(b_0\bar{v}_2(a)-a_0\bar{v}_1(a))u_3\\&-\frac{1}{\sigma_1}(d_0\bar{v}_2(b)-c_0\bar{v}_1(b))u_4\end{aligned} \tag{5}$$

By the boundary condition (4), we have

$$\frac{1}{\sigma_0}(b_0\bar{v}_2(a)-a_0\bar{v}_1(a))u_3-\frac{1}{\sigma_0}(b_0u_2(a)-a_0u_1(a))\bar{v}_3=u_2(a)\bar{v}_1(a)-\bar{v}_2(a)u_1(a), \tag{6}$$

$$\frac{1}{\sigma_1}(d_0u_2(b)-c_0u_1(b))\bar{v}_4-\frac{1}{\sigma_1}(d_0\bar{v}_2(b)-c_0\bar{v}_1(b))u_4=u_1(b)\bar{v}_2(b)-\bar{v}_1(b)u_2(b). \tag{7}$$

Consequently, combining (5)-(7), we obtain

$$\langle \mathcal{L}\boldsymbol{U}, \boldsymbol{V}\rangle = \langle \boldsymbol{U}, \mathcal{L}\boldsymbol{V}\rangle,$$

which implies that operator $\mathcal{L}$ is symmetric.

Since $\mathcal{L}$ is symmetric, it suffices to prove that for any $\boldsymbol{Y}$ and some $\boldsymbol{Z}\in D(\mathcal{L}^*)$, $\boldsymbol{U}\in H$ satisfying $\langle \mathcal{L}\boldsymbol{Y}, \boldsymbol{Z}\rangle = \langle \boldsymbol{Y}, \boldsymbol{U}\rangle$, then $\boldsymbol{Z}\in D(\mathcal{L})$ and $\mathcal{L}\boldsymbol{Z}=\boldsymbol{U}$, where $\boldsymbol{Z}=\left(z_1(x), z_2(x), z_3, z_4\right)^{\mathrm{T}}$, $\boldsymbol{U}=\left(u_1(x), u_2(x), u_3, u_4\right)^{\mathrm{T}}$. It means that $\boldsymbol{Z}$ satisfies following conditions:

(i) $z_i(x)\in \mathrm{AC}[a,b]$, $i=1,2$;

(ii) $z_3=a_1z_1(a)-b_1z_2(a)$, $z_4=c_1z_1(b)-d_1z_2(b)$;

(iii) $u_3=b_0z_2(a)-a_0z_1(a)$, $u_4=d_0z_2(b)-c_0z_1(b)$;

(iv) $\mathcal{L}(z_1,z_2)^{\mathrm{T}}=(u_1,u_2)^{\mathrm{T}}$.

Indeed, the above conditions imply that $D(\mathcal{L}^*)\subset D(\mathcal{L})$.

Step 1 For arbitrary $\boldsymbol{V}=(v_1(x), v_2(x), 0, 0)^{\mathrm{T}}\in C_0^\infty\oplus C_0^\infty\oplus\{0\}\oplus\{0\}$, there is $\langle \mathcal{L}\boldsymbol{V}, \boldsymbol{Z}\rangle = \langle \boldsymbol{V}, \boldsymbol{U}\rangle$. Moreover, we arrive at

$$\int_a^b (v_1(x)\bar{z}_1(x)+v_2(x)\bar{z}_2(x))\,\mathrm{d}x$$
$$=\int_a^b (v_1(x)\bar{u}_1(x)+v_2(x)\bar{u}_2(x))\,\mathrm{d}x.$$

Namely, $\langle \mathcal{L}\boldsymbol{V}, \boldsymbol{Z}\rangle = \langle \boldsymbol{V}, \boldsymbol{U}\rangle$. Since $\mathcal{L}$ is symmetric, combined with $\boldsymbol{V}=\left(v_1(x), v_2(x), 0, 0\right)^{\mathrm{T}}\in C_0^\infty\oplus C_0^\infty\oplus\{0\}\oplus\{0\}$, we can obtain $\langle \mathcal{L}\boldsymbol{V}, \boldsymbol{Z}\rangle = \langle \boldsymbol{V}, \mathcal{L}\boldsymbol{Z}\rangle = \langle \boldsymbol{V}, \boldsymbol{U}\rangle$. In view of the classical theory of differential operators, (i) and (iv) hold.

Step 2 According to (iv) and the relation $\langle \mathcal{L}\boldsymbol{Y}, \boldsymbol{Z}\rangle = \langle \boldsymbol{Y}, \boldsymbol{U}\rangle$, we have

$$\left\langle \boldsymbol{B}\mathcal{U}'+\boldsymbol{Q}(x)\mathcal{U}, \mathcal{Z}\right\rangle - \frac{1}{\sigma_0}(b_0y_2(a)-a_0y_1(a))\bar{z}_3 + \frac{1}{\sigma_1}(d_0y_2(b)-c_0y_1(b))\bar{z}_4$$
$$= \left\langle \mathcal{Y}, \boldsymbol{B}\mathcal{Z}'+\boldsymbol{Q}(x)\mathcal{Z}\right\rangle - \frac{1}{\sigma_0}y_3\bar{z}_3 + \frac{1}{\sigma_1}y_4\bar{z}_4.$$

In light of

$$\left\langle \boldsymbol{B}\mathcal{U}'+\boldsymbol{Q}(x)\mathcal{U}, \mathcal{Z}\right\rangle - \left\langle \mathcal{Y}, \boldsymbol{B}\mathcal{Z}'+\boldsymbol{Q}(x)\mathcal{Z}\right\rangle$$
$$=\bar{z}_1(b)y_2(b)-\bar{z}_2(b)y_1(b)+y_1(a)\bar{z}_2(a)-y_2(a)\bar{z}_1(a).$$

Furthermore, one has

$$\frac{1}{\sigma_1}y_4\bar{z}_4-\frac{1}{\sigma_0}y_3\bar{z}_3+\frac{1}{\sigma_0}(b_0y_2(a)-a_0y_1(a))\bar{z}_3 - \frac{1}{\sigma_1}(d_0y_2(b)-c_0y_1(b))\bar{z}_4$$
$$=\bar{z}_1(b)y_2(b)-\bar{z}_2(b)y_1(b)+y_1(a)\bar{z}_2(a)-y_2(a)\bar{z}_1(a). \quad (8)$$

Using Naimark's patching lemma[16], there exists a $\boldsymbol{Y}=\left(y_1(x), y_2(x), y_3, y_4\right)^{\mathrm{T}}\in D(\mathcal{L})$ such that

$$y_1(b)=y_2(b)=0,\ y_2(a)=-a_1,\ y_1(a)=-b_1.$$

Then by (8), we have $z_3=a_1z_1(a)-b_1z_2(a)$. Similarly, we get

$$y_2(a)=y_1(a)=0,\ y_2(b)=c_1,\ y_1(b)=d_1.$$

Moreover, $z_4=c_1z_1(b)-d_1z_2(b)$. Therefore, (ii) holds. We can prove (iii) by using the similar method, hence we omit the details. The proof is completed.

Corollary 1 The following assertions are true:

1) The two vector eigenfunctions corresponding to different eigenvalues of λ_1 and λ_2 are orthogonal in the following sense

$$\int_a^b (u_1(x)\bar{z}_1(x)+u_2(x)\bar{z}_2(x))\,\mathrm{d}x-\frac{1}{\sigma_0}u_3\bar{z}_3+\frac{1}{\sigma_1}u_4\bar{z}_4=0.$$

2) All eigenvalues of the operator $\mathcal{L}$ are real, and all vector-eigenfunctions are real-valued.

Theorem 2[11] There exists an unboundedly decreasing sequence $\{\lambda_{-n}\}_{n=1}^\infty$ of negative eigenvalues and an unboundedly increasing sequence $\{\lambda_n\}_{n=1}^\infty$ of nonnegative eigenvalues of the boundary value problems (3)-(4):

$$-\infty\leftarrow\cdots<\lambda_{-n}\cdots<\lambda_{-2}<\lambda_{-1}<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots<\rightarrow+\infty.$$

Let $\boldsymbol{\Phi}(x,\lambda)=(\phi_1(x,\lambda),\ \phi_2(x,\lambda))^{\mathrm{T}}$ and $\boldsymbol{\Psi}(x,\lambda)=(\psi_1(x,\lambda), \psi_2(x,\lambda))^{\mathrm{T}}$ be the fundamental solutions of (3), which satisfy the initial condition

$$\phi_1(a,\lambda)=\psi_2(a,\lambda)=1,\ \ \phi_2(a,\lambda)=\psi_1(a,\lambda)=0.$$

Then, $\boldsymbol{\Phi}(x,\lambda)$ and $\boldsymbol{\Psi}(x,\lambda)$ are linearly independent and entire functions of λ.

Denote

$$\boldsymbol{\Lambda}_\lambda(x,\lambda):=\begin{pmatrix}\phi_1(x,\lambda) & \psi_1(x,\lambda)\\ \phi_2(x,\lambda) & \psi_2(x,\lambda)\end{pmatrix}.$$

Lemma 1 $\lambda\in\mathbb{C}$ is an eigenvalue of (3)-(4) if and only if $\Delta(\lambda)=\det\left[\boldsymbol{M}_\lambda+\boldsymbol{N}_\lambda\boldsymbol{\Lambda}_\lambda(b)\right]=0$.

Proof By using similar methods in Ref. [10], we can obtain the assertion holds.

2 Continuous Dependence of Eigenvalues and Eigenfunction

Denote $\boldsymbol{R}=\left[a_0,a_1;b_0,b_1\right]$, $\boldsymbol{G}=\left[c_0,c_1;d_0,d_1\right]$. We introduce Banach space

$$\mathcal{B}:=L_1(a,b)\oplus L_1(a,b)\oplus M_{2\times2}(\mathbb{C})\oplus M_{2\times2}(\mathbb{C})$$

equipped with the norm

$$\|K\|:=\int_a^b(|p|+|q|)\,\mathrm{d}x+\|\boldsymbol{R}\|_{\mathbb{M}}+\|\boldsymbol{G}\|_{\mathbb{M}}.$$

For any $K=(p(x), q(x), \boldsymbol{R}, \boldsymbol{G})\in\mathcal{B}$, where $\|\cdot\|_{\mathbb{M}}$ denotes the matrix normal. Let's construct a boundary condition space $\Omega=\{K\in\mathcal{B}$: the coefficient in (3)-(4), and $\sigma_0>0$, $\sigma_1<0$ hold$\}$. Then Ω is a closed subset of $\mathcal{B}$ and inherits its topology $\mathcal{B}$.

Theorem 3 Let $\tilde{K}=(\tilde{p},\tilde{q},\tilde{\boldsymbol{R}},\tilde{\boldsymbol{G}})\in\Omega$, and $\lambda(\tilde{K})$ be an eigenvalue of spectral problems (3)-(4). Then for any sufficiently small $\varepsilon>0$, there exists $\delta>0$ such that if $\left\|\tilde{K}-K\right\|<\delta$, the spectral problems (3)-(4) have exactly one eigenvalue satisfying $|\lambda(\tilde{K})-\lambda(K)|<\varepsilon$.

Proof It is well known that $\lambda(K)$ is eigenvalue of (3)-(4) if and only if Lemma 1 holds. It is obvious that $\Delta(\lambda(K))$ is not constant with regard to λ since λ is isolated eigenvalue. Furthermore, for any $K\in\Omega$, $\Delta(\lambda(K))$ is an entire function of λ. Hence, there exists $\varepsilon>0$ such that $\Delta(\lambda)\neq 0$ for $\lambda\in\left\{\lambda\in\mathbb{C}:|\lambda-\lambda(\tilde{K})|=\varepsilon\right\}$. By the well known theorem on continuity of the roots of an equation as a function of parameters, see Ref. [17], the assertion holds.

The normalized eigenfunction $\boldsymbol{U}=\left(u_1(x),u_2(x),u_3,u_4\right)^{\mathrm{T}}$ of spectral problems (3)-(4) is defined as follows:

$$\langle \boldsymbol{U},\boldsymbol{U}\rangle=\int_a^b(u_1(x)\bar{u}_1(x)+u_2(x)\bar{u}_2(x))\,\mathrm{d}x-\frac{1}{\sigma_0}u_3\bar{u}_3+\frac{1}{\sigma_1}u_4\bar{u}_4=1.$$

Theorem 4 Let $\lambda(K)$ is an eigenvalue of operator $\mathcal{L}$ with $K\in\Omega$ and $\left(w_1(x),w_2(x),w_3,w_4\right)^{\mathrm{T}}\in H$ is a normalized eigenvector for $\lambda(K)$. Then there exists a normalized eigenvector $\left(v_1(x),v_2(x),v_3,v_4\right)^{\mathrm{T}}\in H$ for $\lambda(\tilde{K})$ with $\tilde{K}\in\Omega$ such that

$$(v_1(x),v_2(x))\to(w_1(x),w_2(x)),\ v_3\to w_3,\ v_4\to w_4$$

as $\tilde{K}\to K$ both uniformly on $[a,b]$.

Proof We know that $\lambda(K)$ is simple. Let $\left(u_1(x,K),u_2(x,K),u_3,u_4\right)^{\mathrm{T}}$ be a normalized eigenvector of operator $\mathcal{L}$. Then in view of inner product in the Hilbert space H, we have

$$\left\|(u_1(x),u_2(x),u_3,u_4)^{\mathrm{T}}\right\|=\int_a^b(u_1\bar{u}_1+u_2\bar{u}_2)\,\mathrm{d}x-\frac{1}{\sigma_0}u_3\bar{u}_3+\frac{1}{\sigma_1}u_4\bar{u}_4=1$$

and $\lambda(K)$ is the corresponding eigenvalue. Theorem 3 means that there exists $\lambda(\tilde{K})$ such that $\lambda(\tilde{K})\to\lambda(K)$ as $\tilde{K}\to K$.

Denote the boundary conditions matrix as follows

$$(\boldsymbol{M}_\lambda,\boldsymbol{N}_\lambda)(K)=\begin{pmatrix}a_1\lambda(K)+a_0 & -(b_1\lambda(K)+b_0) & 0 & 0\\ 0 & 0 & c_1\lambda(K)+c_0 & -(d_1\lambda(K)+d_0)\end{pmatrix}.$$

Then $(\boldsymbol{M}_\lambda,\boldsymbol{N}_\lambda)(\tilde{K})\to(\boldsymbol{M}_\lambda,\boldsymbol{N}_\lambda)(K)$ as $\tilde{K}\to K$. Besides, there exists $\left(u_1(x,\tilde{K}),u_2(x,\tilde{K}),u_3(\tilde{K}),u_4(\tilde{K})\right)^{\mathrm{T}}$ satisfying the $(\boldsymbol{M}_\lambda,\boldsymbol{N}_\lambda)(\tilde{K})$ for $\lambda(\tilde{K})$ and $\left\|\left(u_1(x,\tilde{K}),u_2(x,\tilde{K}),u_3(\tilde{K}),u_4(\tilde{K})\right)^{\mathrm{T}}\right\|=1$. Therefore, we obtain

$$(u_1(x,\tilde{K}),u_2(x,\tilde{K}))\to(u_1(x,K),u_2(x,K)),\ u_3(\tilde{K})\to u_3(K),\ u_4(\tilde{K})\to u_4(K)$$

as $\tilde{K}\to K$ both uniformly on $[a,b]$. Let

$$(v_1(x),v_2(x),v_3,v_4)^{\mathrm{T}}=\frac{(u_1(x,\tilde{K}),u_2(x,\tilde{K}),u_3(\tilde{K}),u_4(\tilde{K}))^{\mathrm{T}}}{\left\|\left(u_1(x,\tilde{K}),u_2(x,\tilde{K}),u_3(\tilde{K}),u_4(\tilde{K})\right)^{\mathrm{T}}\right\|},$$

$$(w_1(x),w_2(x),w_3,w_4)^{\mathrm{T}}=\frac{(u_1(x,K),u_2(x,K),u_3(K),u_4(K))^{\mathrm{T}}}{\left\|(u_1(x,K),u_2(x,K),u_3(K),u_4(K))^{\mathrm{T}}\right\|}.$$

Then the desired assertion holds. The proof is completed.

3 Differentiability of Eigenvalue

In this section, we will prove that the simple eigenvalue branch is differentiable for all parameters and obtain the differential expression for all parameter in the sense of Fréchet derivative.

Definition 1[17] A map $\mathcal{T}$ from an open set D of the Banach space E_1 into the Banach space E_2 is Fréchet differentiable at a point $x_0\in D$ if there exists a bounded linear operator $\mathrm{d}\mathcal{T}_{x_0}(s)$: $E_1\to E_2$ such that in some neighborhood of the x_0,

$$\left\|\mathcal{T}(x_0+s)-\mathcal{T}(x_0)-\mathrm{d}\mathcal{T}_{x_0}(s)\right\|=o(\|s\|)\quad\text{as}\quad s\to 0.$$

Theorem 5 Let $\lambda(K)$ be an eigenvalue for the operator $\mathcal{L}$ for $K\in\Omega$ and $(u_1(x),u_2(x),u_3,u_4)^{\mathrm{T}}$ be a normalized eigenvector of $\lambda(K)$. Then λ is Fréchet differentiable with respect to all parameters in K. Specifically, the derivative formulas of λ are given as follows:

Result 1 Fix all components of K except the boundary matrix $\boldsymbol{R}$ and let $\lambda(\boldsymbol{R})=\lambda(K)$ denote the eigenvalue. Then

$$\mathrm{d}\lambda_{\boldsymbol{R}}(\boldsymbol{T})=(-u_1(a),u_2(a))\left[\boldsymbol{I}-\boldsymbol{R}(\boldsymbol{R}+\boldsymbol{T})^{-1}\right]\begin{pmatrix}\bar{u}_2(a)\\ \bar{u}_1(a)\end{pmatrix}$$

for all $\boldsymbol{T}$, where $\det(\boldsymbol{R}+\boldsymbol{T})=\det\boldsymbol{R}=\sigma_0$.

Result 2 Fix all components of K except the boundary matrix $\boldsymbol{G}$ and let $\lambda(\boldsymbol{G})=\lambda(K)$ denote the eigenvalue. Then $\mathrm{d}\lambda_{\boldsymbol{G}}(\boldsymbol{T})=(u_1(b),-u_2(b))\left[\boldsymbol{I}-\boldsymbol{G}(\boldsymbol{G}+\boldsymbol{T})^{-1}\right]\begin{pmatrix}\bar{u}_2(b)\\ \bar{u}_1(b)\end{pmatrix}$, for all $\boldsymbol{T}$, where $\det(\boldsymbol{G}+\boldsymbol{T})=\det\boldsymbol{G}=\sigma_1$.

Result 3 Fix all components of K except p or q and let $\lambda(p)=\lambda(K)$ or $\lambda(q)=\lambda(K)$ denote the eigenvalue. Then

$$\mathrm{d}\lambda_p(s)=\int_a^b 2\mathrm{Re}(u_1\bar{u}_2)s\,\mathrm{d}x,\quad \text{or}$$

$$\mathrm{d}\lambda_q(s)=\int_a^b 2\mathrm{Re}(u_1\bar{u}_2)s\,\mathrm{d}x,\quad s\in L_1(a,b).$$

Result 4 Fix all components of K except $\boldsymbol{Q}$, and let $\lambda(\boldsymbol{Q})=\lambda(K)$ denote the eigenvalue. Then

$$\mathrm{d}\lambda_{\boldsymbol{Q}}(\boldsymbol{P})=\int_a^b \mathcal{U}^*\boldsymbol{P}\mathcal{U}\mathrm{d}x,\ \text{where } \boldsymbol{P}\in M_{2\times2}(\mathbb{R})$$

such that $\boldsymbol{Q}+\boldsymbol{P}$ is symmetric and every element in $\boldsymbol{Q}+\boldsymbol{P}$ belong to $L_1(a,b)$.

Proof 1) Let $\lambda(\boldsymbol{R}+\boldsymbol{T})$, $(v_1(x),v_2(x),v_3,v_4)^{\mathrm{T}}$ denote the eigenvalue and corresponding normalize eigenvector. Direct computation yields

$$\lambda(\boldsymbol{R}+\boldsymbol{T})\mathcal{V}=\boldsymbol{B}(\mathcal{V})'+\boldsymbol{Q}\mathcal{V},\tag{9}$$

$$\lambda(\boldsymbol{R})\mathcal{U}=\boldsymbol{B}\mathcal{U}+\boldsymbol{Q}\mathcal{U}.\tag{10}$$

By (9)-(10) and integration by parts, we obtain

$$\left[\lambda(\boldsymbol{R}+\boldsymbol{T})-\lambda(\boldsymbol{R})\right]\int_a^b \mathcal{V}\mathcal{U}\mathrm{d}x=-\bar{v}_1(b)u_2(b)+\bar{v}_2(b)u_1(b)+u_2(a)\bar{v}_1(a)-u_1(a)\bar{v}_2(a).\tag{11}$$

Let

$$\boldsymbol{R}+\boldsymbol{T}=\begin{pmatrix}\tilde{a}_0 & \tilde{a}_1\\ \tilde{b}_0 & \tilde{b}_1\end{pmatrix}.$$

By (4), we have

$$\left[\lambda(\boldsymbol{R}+\boldsymbol{T})-\lambda(\boldsymbol{R})\right]\frac{1}{\sigma_0}u_3\bar{v}_3=(u_1(a),-u_2(a))\frac{1}{\sigma_0}\begin{pmatrix}a_1\tilde{b}_0-a_0\tilde{b}_1 & a_0\tilde{a}_1-a_1\tilde{a}_0\\ b_1\tilde{b}_0-b_0\tilde{b}_1 & b_0\tilde{a}_1-b_1\tilde{a}_0\end{pmatrix}\begin{pmatrix}\bar{v}_2(a)\\ \bar{v}_1(a)\end{pmatrix}=(u_1(a),-u_2(a))\left[-\boldsymbol{R}(\boldsymbol{R}+\boldsymbol{T})^{-1}\right]\begin{pmatrix}\bar{v}_2(a)\\ \bar{v}_1(a)\end{pmatrix}.\tag{12}$$

Using the similar method, we can obtain

$$\left[\lambda(\boldsymbol{R}+\boldsymbol{T})-\lambda(\boldsymbol{R})\right]\frac{1}{\sigma_1}u_4\bar{v}_4=u_2(b)\bar{v}_1(b)-\bar{v}_2(b)u_1(b).\tag{13}$$

Combining (11)-(13), we have

$$\left[\lambda(\boldsymbol{R}+\boldsymbol{T})-\lambda(\boldsymbol{R})\right]\left[\int_a^b \mathcal{V}\mathcal{U}\mathrm{d}x-\frac{1}{\sigma_0}u_3\bar{v}_3+\frac{1}{\sigma_1}u_4\bar{v}_4\right]=(-u_1(a),u_2(a))\left[\boldsymbol{I}-\boldsymbol{R}(\boldsymbol{R}+\boldsymbol{T})^{-1}\right]\begin{pmatrix}\bar{u}_2(a)\\ \bar{u}_1(a)\end{pmatrix}.$$

Let $\boldsymbol{T}\to 0$, then the desired Result 1 can be obtained by Theorem 4.

2) Using the similar method, we can obtain Result 2.

3) Fix all components of K except p (the situation for q is similar). For $s\in L_1(a,b)$. $(v_1(x),v_2(x),v_3,v_4)^{\mathrm{T}}$ denote eigenvalue and corresponding normalized eigenvector. Let

$$\tilde{\boldsymbol{Q}}(x)=\begin{pmatrix}p(x)+s(x) & q(x)\\ q(x) & p(x)+s(x)\end{pmatrix}.$$

By (9) and (10), we have

$$\left[\lambda(p+s)-\lambda(p)\right]\int_a^b \mathcal{V}\mathcal{U}\,\mathrm{d}x=-\bar{v}_1(b)u_2(b)+\bar{v}_2(b)u_1(b)+u_2(a)\bar{v}_1(a)-u_1(a)\bar{v}_2(a)-\int_a^b \mathcal{V}(\boldsymbol{Q}-\tilde{\boldsymbol{Q}})\mathcal{U}\mathrm{d}x.$$

By (4), we obtain $\left[\lambda(p+s)-\lambda(p)\right]\frac{1}{\sigma_0}u_3\bar{v}_3=u_2(a)\bar{v}_1(a)-u_1(a)\bar{v}_2(a)$.

Similarly, $\left[\lambda(p+s)-\lambda(p)\right]\frac{1}{\sigma_1}u_4\bar{v}_4=\bar{v}_1(b)u_2(b)-\bar{v}_2(b)u_1(b)$.

Hence

$$\left[\lambda(p+s)-\lambda(p)\right]\int_a^b \mathcal{V}\mathcal{U}\mathrm{d}x=\int_a^b(\bar{v}_1,\bar{v}_2)\begin{pmatrix} s & 0\\ 0 & s\end{pmatrix}\begin{pmatrix} u_1\\ u_2\end{pmatrix}\mathrm{d}x=\int_a^b 2\mathrm{Re}(u_1\bar{u}_2)s\,\mathrm{d}x.$$

Then the desired Result 3 can be obtained.

4) In the same way, we can obtain Result 4. The proof is completed.

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边界条件含有谱参数的Dirac算子的谱性质

仲林路， 杜高峰†

西北师范大学 数学与统计学院，甘肃 兰州 730070

摘要：本文主要研究边界条件含有谱参数的Dirac算子特征值问题的一些谱性质。首先，通过建立适当的Hilbert空间将谱问题转化为线性算子特征值问题，并推导出了自伴算子的一些重要性质。其次，通过建立边界条件空间，构造嵌入映射，证明了Dirac算子的简单特征值分支不仅是连续的，而且是光滑的。最后，在Fréchet导数意义下，我们得到了特征值分支关于所有参数的微分表达式。

关键词：特征参数依赖边界条件；谱性质；Dirac算子

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