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The Global Attractor of 2D g -Navier-Stokes Equations with Damping and Delay

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Abstract: In this article, the global attractors of 2D g -Navier-Stokes equations are obtained in the space of C_{H^s} and C_{V^s} respectively. When the external force f is sufficiently small, the studies indicate that the global attractor in C_{H^s} is equal to the global attractor in C_{V^s} .

Key words: global attractor; g -Navier-Stokes equation; damping; delay

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0 Introduction

The Navier-Stokes equations (NSE) are important in fluid mechanics and turbulence. In the last decades, the research on the asymptotic properties of the solution for NSE has attracted the attention of many scholars^[1-7]. Especially in the past years, the NSE with nonlinear damping and delay has been studied^[8-16], where the damping comes from the resistance of the flow or the friction effect. The delays denote some type of external forces that can be applied to control one system, and these forces can be regarded as the present state of the system or its history.

In recent years, the research on 2D g -Navier-Stokes equations (g NSE) has been paid more attention by scholars. It is derived from 3D NSE by the vertical mean operator in Ref. [17], and its form is as follows:

$$\begin{cases} \frac{\partial u}{\partial t} - \mu \Delta u + (u \cdot \nabla)u + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot gu = 0 & \text{in } \Omega, \end{cases} \quad (1)$$

where $g = g(x_1, x_2)$ is a suitable smooth real-valued function defined on $(x_1, x_2) \in \Omega$ and Ω is the bounded domain in \mathbb{R}^2 . We study the 2D g NSE as a small perturbation of the usual NSE, so we want to understand the NSE completely by studying the 2D g NSE systematically. Therefore, the research on the g NSE has a theoretical basis and practical significance.

There are many researches on g -Navier-Stokes equations recently^[18-26]. In Ref. [18], the well-posedness of solutions for g NSE was proved on \mathbb{R}^2 for $n = 2, 3$. In Ref. [19], Roh obtained the existence of the global attractors of g NSE and proved that the semiflows were robust to g . Moreover, the existence of global solutions and the global attractor of g NSE were proved, and the dimension of the global attractor was estimated in Ref.

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[20]. Also, the global attractor of gNSE with linear dampness on \mathbb{R}^2 were proved, and the estimation of dimensions was also obtained in Ref. [21]. We investigated the existence of a pullback attractor for the 2D non-autonomous gNSE on the bounded domains in Ref. [22]. Quyet proved the existence of a pullback attractor in V_g for the process in Ref. [23]. Recently, we have discussed the uniform attractor and pullback attractor of gNSE with damping and time delay in Refs. [24-26], and obtained the uniform attractor and pullback attractor using the method of the energy equation and the pullback condition, respectively.

It is well-known that the research of global attractors is an important problem in infinite-dimensional dynamical systems (IDDS). In addition, the dimension is a very important property of the global attractor. However, as far as we know, the global attractor of 2D autonomous gNSE with damping and delay has been studied rarely. Therefore, in this article, we research global attractor of the 2D autonomous gNSE, which has nonlinear damping and time delay on some bounded domain $\Omega \subset \mathbb{R}^2$, and its usual form is as follows:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + c|u|^{\beta-1}u + \nabla p = f(x, t) + h(t, u_t) & \text{in } \Omega \times (0, \infty), \\ \nabla \cdot gu = 0 & \text{in } \Omega \times (0, \infty), \\ u(x, t) = 0 & \text{on } \partial\Omega, \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (2)$$

where $p(x, t) \in \mathbb{R}$ denotes the pressure and $u(x, t) \in \mathbb{R}^2$ as the velocity, $\nu > 0$ and $c|u|^{\beta-1}u$ are nonlinear dampings, $\beta \geq 1$ and $c > 0$ are constants, $0 < m_0 \leq g = g(x_1, x_2) \leq M_0$, where $g = g(x_1, x_2)$ is a real-valued smooth function. $f = f(x, t)$ is the external force, $h = h(t, u_t)$ is another external force term with time delay, u_t is the function which can be defined by $u_t(\theta) = u(t + \theta)$. $\forall \theta \in (-r, 0), r > 0$ is a constant.

Definition 1^[26] Let $C_{H_g} = C^0([-h, 0]; H_g)$, $C_{V_g} = C^0([-h, 0]; V_g)$, $h: \mathbb{R} \times C_{H_g} \rightarrow (L^2(\Omega))^2$ satisfies the following assumptions:

- (I) $h(0) = 0$;
- (II) there exists $L_g > 0$, such that $\forall \zeta, \eta \in C_{H_g}, |h(\zeta) - h(\eta)| \leq L_g \|\zeta - \eta\|_{C_{H_g}}$;
- (III) there exists $C_g > 0$, such that $\forall t \in [0, T], u, v \in C(-r, T; H_g)$,

$$\int_0^t |h(u_s) - h(v_s)| ds \leq C_g \int_{-r}^t |u(s) - v(s)|^2 ds.$$

Definition 2^[26] Let $u_0 \in H_g, f \in L^2(\tau, T; V'_g)$, $h: \mathbb{R} \times$

$C_{H_g} \rightarrow (L^2(\Omega))^2$ satisfies (I)-(III).

For any $\tau \in \mathbb{R}, u \in L^\infty(\tau, T; V_g) \cap L^2(\tau, T; V_g) \cap L^{\beta+1}(\tau, T; L^{\beta+1}(\Omega)), \forall T > \tau$ is called a weak solution of (2) if it fulfills

$$\begin{aligned} & \frac{d}{dt} u(t) + \nu A_g u(t) + B(u(t)) + c|u|^{\beta-1}u + \nu R(u(t)) \\ & = f(x, t) + h(t, u_t) \text{ on } D'(\tau, +\infty; V'_g), u(\tau) = u_0. \end{aligned}$$

Theorem 1^[26] Let $\beta \geq 1, f \in L^2(\tau, T; V'_g)$, $h: \mathbb{R} \times C_{H_g} \rightarrow (L^2(\Omega))^2$ satisfies (I)-(III), then for every $u_\tau \in V_g$, equation (2) has the only weak solution $u(t) = u(t; \tau, u_\tau) \in L^\infty(\tau, T; V_g) \cap L^2(\tau, T; V_g) \cap L^{\beta+1}(\tau, T; L^{\beta+1}(\Omega))$, and $u(t)$ continuously depends on the initial value in V_g .

The main results of this article are as follows:

Lemma 1 For any $\phi \in H_g, f \in (L^2(\Omega))^2$, let $h: C_{H_g} \rightarrow (L^2(\Omega))^2$ and satisfy (I) - (III), then $S_\tau = \{\phi \in C_{H_g}; \|\phi\|_{C_{H_g}} \leq \tau k = \frac{\tau \|f\|}{\lambda - L_g}\}$ is positive invariant set of $S(t)$,

where τ is a constant and $\tau > 1$.

Lemma 2 For any $\phi \in H_g, f \in (L^2(\Omega))^2$, let $h: C_{H_g} \rightarrow (L^2(\Omega))^2$ and satisfy (I) - (III), then $S = \{\phi \in C_{H_g}; \|\phi\|_{C_{H_g}} \leq k = \frac{\|f\|}{\lambda - L_g}\}$ is a global absorbed set of $S(t)$ in C_{H_g} .

Lemma 3 For any $\phi \in H_g, f \in (L^2(\Omega))^2$, let $h: C_{H_g} \rightarrow (L^2(\Omega))^2$ and satisfy (I) - (III), and $|\nabla g|_\infty^2 < \frac{\lambda_1 m_0^2}{4}$, then $\Gamma = \{\phi \in C_{V_g}; \|\phi\|_{C_{V_g}} \leq \rho\}$ is a global absorbed set of $S(t)$ in C_{V_g} , where ρ is a given constant.

Theorem 2 For any $\phi \in H_g, f \in (L^2(\Omega))^2$, let $h: C_{H_g} \rightarrow (L^2(\Omega))^2$ and satisfy (I)-(III), then equation (2) possesses a global attractor A_0 in C_{H_g} , which is compact and connected in C_{H_g} , and it attracts any bounded set in C_{H_g} .

Theorem 3 Let $f \in H_g$, then the semigroup $S(t)$ corresponding to equation (2) has a global attractor A_1 in C_{V_g} satisfying

- 1) A_1 is compact in C_{V_g} ;
 - 2) $S(t)A_1 = A_1$;
 - 3) For any bounded subset B_1 of C_{V_g} ,
- $$\lim_{t \rightarrow \infty} \text{dist}_{V_g}(S(t)B_1, A_1) = \lim_{t \rightarrow \infty} \sup_{\zeta \rightarrow B_1} \text{dist}_{V_g}(S(t)\zeta, A_1) = 0.$$

Lemma 4 Let $f \in H_g$ and f be sufficiently small, B is bounded subset in H_g . Let $u(t) = S(t)u_0$ be a corresponding solution to equation (2), $u_0 \in B$, then there exist t_0 and constant $M > 0$, such that $\|u(t)\| = \|S(t)u_0\| \leq M, \forall u_0 \in B \subset H_g, \forall t \geq t_0$.

Theorem 4 Let $f \in H_g$ and f be sufficiently small, then $A_0 = A_1$, where A_0 and A_1 denote global attractors in

C_{H_g} and C_{V_g} , respectively.

This article is organized as follows. In Section 1, we give some results on the classical theory of the global attractor. In Section 2, we obtain the global attractor of 2D g NSE with damping and time delay. In Section 3, we give some relevant conclusions.

1 Preliminaries

We define $L^2(g)=(L^2(\Omega))^2$ and $H_0^1(g)=(H_0^1(\Omega))^2$, the inner product of $L^2(g)$ is $(u, v)=\int_{\Omega} u \cdot v g dx$ and inner product of $H_0^1(g)$ is $((u, v))=\int_{\Omega} \sum_{j=1}^2 \nabla u_j \cdot \nabla v_j g dx$, corresponding norm is $|\cdot|=(\cdot, \cdot)^{1/2}$ and $\|\cdot\|=((\cdot, \cdot))^{1/2}$ respectively.

Let $M=\{v \in (D(\Omega))^2: \nabla \cdot g v=0 \text{ in } \Omega\}$ and $D(\Omega)$ is the space of C^∞ functions that have compact support contained in $\Omega \subset \mathbb{R}^2$. H_g is the closure of M in $L^2(g)$ which is endowed with the inner product and norm of $L^2(g)$, V_g is the closure of M in $H_0^1(g)$ endowed with the inner product and norm of $H_0^1(g)$.

Let $\lambda_1 > 0$, we have

$$\int_{\Omega} \phi^2 g dx \leq \frac{1}{\lambda_1} \int_{\Omega} |\nabla \phi|^2 g dx, \quad \forall \phi \in H_0^1(\Omega), \tag{3}$$

then

$$|u|^2 \leq \frac{1}{\lambda_1} \|u\|^2, \quad \forall u \in V_g. \tag{4}$$

The g -Laplacian operator is defined as follows:

$$-\Delta_g u = -\frac{(\nabla \cdot g \nabla) u}{g} = -\Delta u - \frac{1}{g} \nabla g \cdot \nabla u.$$

The first formula in (2) can be expressed as follows:

$$\frac{\partial u}{\partial t} - \nu \Delta_g u + \nu \frac{\nabla g}{g} \cdot \nabla u + (u, \nabla) u + c|u|^{\beta-1} u + \nabla p = f + h(t, u). \tag{5}$$

The g -orthogonal projection is defined by $P_g: L^2(g) \rightarrow H_g$ and g -Stokes operator is defined by $A_g u = -P_g(\frac{1}{g}(\nabla \cdot (g \nabla u)))$. Applying the projection P_g to (2), for $\forall v \in V_g, \forall t > 0$, we obtain

$$\begin{aligned} & \frac{d}{dt} (u, v) + \nu((u, v)) + b_g(u, u, v) + c(|u|^{\beta-1} u, v) + \nu(Ru, v) \\ & = \langle f, v \rangle + \langle h(t, u), v \rangle, \end{aligned} \tag{6}$$

$$u(0) = u_0, \tag{7}$$

where $b_g: V_g \times V_g \times V_g \rightarrow \mathbb{R}$ and $b_g(u, v, w) = \sum_{i,j=1}^2 \int_{\Omega} u_i \frac{\partial v_j}{\partial x} w_j g dx$

$Ru = P_g[\frac{1}{g}(\nabla g \cdot \nabla)u], \forall u \in V_g$. We define $G(u) = P_g F(u)$ and $F(u) = c|u|^{\beta-1}u$, then the formulations (6) and (7) are equivalent to the following equations

$$\frac{du}{dt} + \nu A_g u + Bu + G(u) + \nu Ru = f + h, \tag{8}$$

$$u(0) = u_0, \tag{9}$$

where $A_g: V_g \rightarrow V_g'$ for $\forall u, v \in V_g$, we have $\langle A_g u, v \rangle = ((u, v))$. $B(u) = B(u, u) = P_g(u \cdot \nabla)u$ is a bilinear operator, and $B: V_g \times V_g \rightarrow V_g'$ with $\langle B(u, v), w \rangle = b_g(u, v, w), \forall u, v, w \in V_g$.

For any $u, v \in D(A_g)$, we have $|B(u, v)| \leq C|u|^{1/2}|A_g u|^{1/2}|v|$, where C denotes positive constant. From Refs. [10, 17, 19, 27], we have the following inequality:

$$|\phi|_{L^{\infty}(\Omega)} \leq C\|\phi\|(1 + \ln \frac{|A_g \phi|^2}{\lambda_1 \|\phi\|^2})^{1/2}, \quad \forall \phi \in D(A_g), \tag{10}$$

$$|B(u, v)| \leq (u \cdot \nabla)v \leq |u|_{L^{\infty}(\Omega)}|\nabla v|, \tag{11}$$

$$|B(u, v)| \leq C\|u\|\|v\|(1 + \ln \frac{|A_g u|^2}{\lambda_1 \|u\|^2})^{1/2}, \tag{12}$$

$$\|B(u)\|_{V_g'} \leq c\|u\|\|u\|, \quad \|Ru\|_{V_g'} \leq \frac{|\nabla g|_{\infty}}{m_0 \lambda_1^{1/2}}\|u\|, \quad \forall u \in V_g. \tag{13}$$

Definition 3^[28] Let M be a complete metric space, a parameter family $S(t), t \geq 0$ of maps $S(t): M \rightarrow M, t \geq 0$ is called C^0 semigroup if

- 1) $S(0)$ is the identity map on M ,
- 2) $S(t+s) = S(t)S(s)$ for all $t, s \geq 0$,
- 3) the function $S(t)x$ is continuous at each point $(t, x) \in [0, \infty) \times M$.

Definition 4^[28] Let $S(t), t \geq 0$ be a C^0 semigroup in a complete metric space M .

A subset B_0 of M is called an absorbing set in M , if for any bounded subset B of M , there exists some $t_1 \geq 0$, such that $S(t)B \subset B_0$, for all $t \geq t_1$.

Definition 5^[28] Let $S(t), t \geq 0$ be a C^0 semigroup in a complete metric space M .

A subset A of M is called a global attractor for the semigroup if A is compact and enjoys the following properties:

- 1) A is an invariant set, i.e., $S(t)A = A$ for any $t \geq 0$;
- 2) A attracts all bounded sets of M . That is, for any bounded subset B of $M, t \rightarrow \infty, d(S(t)B, A) \rightarrow 0$, where $d(B, A)$ is the semidistance of two sets B and $A: d(B, A) = \sup_{x \in B} \inf_{y \in A} d(x, y)$.

Theorem 5^[28] If $S(t)$ is dissipative and B is a compact absorbing set, then there exists a global attractor $A = \omega(B) = \bigcup_{t \geq 0} S(t)B$.

2 Proofs of the Main Results

In this section, we will obtain the global attractors

of 2D gNSE in C_{H_g} and C_{V_g} respectively.

Proof of Lemma 1 Taking the inner product of

$$\frac{du}{dt} + \nu A_g u + B(u) + G(u) + \nu Ru = f + h(t, u_t)$$

with u , we have

$$\begin{aligned} \frac{1}{2} \frac{d|u|^2}{dt} + \nu \|u\|^2 + (Bu, u) + (G(u), u) + \nu (Ru, u) \\ = (f, u) + (h(t, u_t), u), \end{aligned}$$

that is

$$\frac{d|u|^2}{dt} + 2(G(u), u) = 2(f, u) - 2\nu(Ru, u) - 2\nu\|u\|^2 + 2(h(u, u_t), u).$$

Since $2(G(u), u) = 2c|u|^{\beta-1}|u|^2 = 2c|u|^{\beta+1} > 0, (\beta \geq 1, c > 0)$,

$$\begin{aligned} \frac{d|u|^2}{dt} &\leq 2|f| \cdot |u| + 2\nu|Ru| \cdot |u| - 2\nu\|u\|^2 + 2|h(t, u_t)| \cdot |u| \\ &\leq 2|u| \cdot (|f| + |h(t, u_t)|) - 2\nu\|u\|^2 + 2\nu \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} \|u\|^2 \\ &\leq 2|u| \cdot (|f| + |h(t, u_t)|) - 2\nu \lambda_1 (1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}) \|u\|^2. \end{aligned}$$

Let $\lambda = \nu \lambda_1 (1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}})$, then

$$\frac{d|u|^2}{dt} + 2\lambda|u|^2 \leq 2|u|(|f| + |h(t, u_t)|).$$

For any $t \geq 0$, by the Growall inequality, we obtain

$$\begin{aligned} |u|^2 &\leq \|\phi\|_{C_{H_g}}^2 e^{-2\lambda t} + 2 \int_0^t e^{-2\lambda(t-s)} [|u(s)|(|f| + |h(s, u_s)|)] ds \\ &\leq \|\phi\|_{C_{H_g}}^2 e^{-2\lambda t} + 2\lambda \int_0^t e^{-2\lambda(t-s)} [\frac{L_g \|u_s\|_{C_{H_g}}}{\lambda} + \frac{|f|}{\lambda}] |u(s)| ds. \end{aligned}$$

For $\forall t \geq 0$, in the following we will prove $\|u\|_{C_{H_g}} < \tau k$ when $\|\phi\|_{C_{H_g}} < \tau k$.

We do this by contradiction. Suppose there exists $t_1 > 0$, such that $|u(t_1, x)| = \tau k$. When $t < t_1$, we have $|u(t, x)| < \tau k$, that is, for any $0 \leq t \leq t_1$, we have $|u(t, x)| \leq \tau k$.

It can be obtained from the above formula that for any $t \geq 0$, we deduce

$$|u(t_1, x)|^2 \leq \tau^2 k^2 e^{-2\lambda t_1} + 2\lambda \int_0^{t_1} e^{-2\lambda(t_1-s)} \tau k (\frac{L_g \tau k}{\lambda} + \frac{|f|}{\lambda}) ds.$$

Let $k = \frac{|f|}{\lambda - L_g}$, there is $k = \frac{L_g k}{\lambda} + \frac{|f|}{\lambda}$. Then

$$\begin{aligned} |u(t, x)|^2 &\leq \tau^2 k (\frac{L_g k}{\lambda} + \frac{|f|}{\lambda}) e^{-2\lambda t} + \tau k (\frac{\tau k L_g}{\lambda} + \frac{|f|}{\lambda}) 2\lambda e^{-2\lambda t} \int_0^{t_1} e^{2\lambda s} ds \\ &\leq \tau^2 k (\frac{L_g k}{\lambda} + \frac{|f|}{\lambda}) e^{-2\lambda t} + \tau k (\frac{\tau k L_g}{\lambda} + \frac{|f|}{\lambda}) 2\lambda e^{-2\lambda t} \frac{e^{2\lambda t_1} - 1}{2\lambda} \\ &= \tau^2 k (\frac{L_g k}{\lambda} + \frac{|f|}{\lambda}) e^{-2\lambda t} + \tau k (\frac{\tau k L_g}{\lambda} + \frac{|f|}{\lambda}) (1 - e^{-2\lambda t_1}) \\ &\leq \tau(\tau - 1)k \frac{|f|}{\lambda} + \tau k (\frac{\tau k L_g}{\lambda} + \frac{|f|}{\lambda}) = \tau^2 k^2. \end{aligned}$$

This is a contradiction, so S_t is a positive invariant set.

Proof of Lemma 2 For any $\phi \in C_{H_g}$, there exists

$\tau \geq 1$, such that $\phi \in S_\tau$. From Lemma 1, we have $|u(t, x)| \leq \tau k$, so there exists $\sigma > 0$, such that $\limsup_{t \rightarrow \infty} |u| = \sigma$.

For arbitrarily small $\varepsilon > 0$, there exists $t_2 \geq 0$, such that $|u| \leq (\sigma + \varepsilon), t \geq t_2$.

For $\lambda > 0$, for the above ε and k , we have $T > 0$ and $\tau^2 k^2 e^{-2\lambda T} + 2\lambda \int_T^\infty e^{-2\lambda s} \tau k (\frac{L_g \tau k}{\lambda} + \frac{|f|}{\lambda}) ds \leq \varepsilon$.

Hence, as $t \geq T + t_2$,

$$\begin{aligned} |u|^2 &\leq \|\phi\|_{C_{H_g}}^2 e^{-2\lambda t} + 2\lambda \{ \int_0^{t-T} + \int_{t-T}^t \} e^{-2\lambda(t-s)} |u(s)| (\frac{L_g}{\lambda} \|u\|_{C_{H_g}} + \frac{|f|}{\lambda}) ds \\ &\leq \varepsilon + 2\lambda(\sigma + \varepsilon) \int_{t-T}^t e^{-2\lambda(t-s)} (\frac{L_g}{\lambda} \|u\|_{C_{H_g}} + \frac{|f|}{\lambda}) ds \\ &\leq \varepsilon + (\sigma + \varepsilon) [\frac{L_g}{\lambda} (\sigma + \varepsilon) + \frac{|f|}{\lambda}]. \end{aligned}$$

Let $\varepsilon \rightarrow 0$, then $\sigma^2 = \sigma (\frac{\sigma L_g}{\lambda} + \frac{|f|}{\lambda})$, therefore $\sigma \leq k = \frac{|f|}{\lambda - L_g}$, that is, $\limsup_{t \rightarrow \infty} |u| \leq k = \frac{|f|}{\lambda - L_g}$, so S is the global absorbed set of the semigroup $S(t)$ in C_{H_g} .

Proof of Lemma 3 Taking the inner product of

(8) with $A_g u$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu |A_g u|^2 + b(u, u, A_g u) + (G(u), A_g u) + \nu (Ru, A_g u) \\ = (f, A_g u) + (h(t, u_t), A_g u). \end{aligned}$$

Since $|b(u, u, A_g u)| \leq c_1 |u|^2 \|u\| |A_g u| \leq \frac{\nu}{4} |A_g u|^2 + \frac{c_1'}{\nu^3} |u|^2 \|u\|^4$,

where $c_1' = \frac{27c_1^4}{4}$,

$$\begin{aligned} |(G(u), A_g u)| &\leq c|u|^{\beta-1}|u||A_g u| \leq c(\frac{|u|^{2\beta}}{2\nu} + \frac{\nu}{2}|A_g u|^2). \\ |(f, A_g u) + (h(t, u_t), A_g u)| &\leq |A_g u|(|f| + |h(t, u_t)|) \\ &\leq \frac{\nu}{8}|A_g u|^2 + \frac{4}{\nu}(|f|^2 + L_g^2 \|u\|_{C_{H_g}}^2). \\ \nu|(Ru, A_g u)| &\leq \nu|Ru| \cdot |A_g u| \leq \frac{\nu}{8}|A_g u|^2 + 2\nu|Ru|^2 \\ &\leq \frac{\nu}{8}|A_g u|^2 + 2\nu \frac{|\nabla g|_\infty^2}{m_0^2 \lambda_1} \|u\|^2. \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dt} \|u\|^2 + 2\nu |A_g u|^2 \\ \leq -2b(u, u, A_g u) - 2(G(u), A_g u) - 2\nu(Ru, A_g u) + 2(h(t, u_t), A_g u) \\ \leq \frac{\nu}{2}|A_g u|^2 + \frac{2c_1'}{\nu^3} |u|^2 \|u\|^4 + \frac{c|u|^{2\beta}}{2\nu} + \frac{c\nu}{2}|A_g u|^2 + \frac{\nu}{4}|A_g u|^2 \\ + \frac{4\nu|\nabla g|_\infty^2}{m_0^2 \lambda_1} \|u\|^2 + \frac{\nu}{4}|A_g u|^2 + \frac{8}{\nu}(|f|^2 + L_g^2 \|u\|_{L_{H_g}}^2) \\ \leq (\nu + \frac{c\nu}{2})|A_g u|^2 + \frac{c|u|^{2\beta}}{2\nu} + \frac{2c_1'}{\nu^3} |u|^2 \|u\|^4 \\ + \frac{4\nu|\nabla g|_\infty^2}{m_0^2 \lambda_1} \|u\|^2 + \frac{8}{\nu}(|f|^2 + L_g^2 \|u\|_{L_{H_g}}^2), \end{aligned}$$

we have

$$\begin{aligned} \frac{d}{dt} \|u\|^2 + \lambda_1 \left(v - \frac{cv}{2} \right) \|u\|^2 &\leq \frac{c|u|^{2\beta}}{2v} + \frac{4v|\nabla g|_\infty^2}{m_0^2} \|u\|^2 \\ &+ \frac{2c'_1}{v^3} |u|^2 \|u\|^4 + \frac{8}{v} (|f|^2 + L_g^2 \|u\|_{L_{u_g}}^2). \end{aligned} \tag{14}$$

For any $\phi \in C_{H_g}$, from Lemma 1, there exists $\tau > 0$, such that $\phi \in S_\tau$, that is, for any $\varepsilon \geq 0$, we have $|u| \leq \tau k$. From (8) and $|A_g u|^2 \leq \lambda_m \|u\|^2$,

$$\begin{aligned} \frac{d}{dt} |u|^2 &\leq -2v \left(1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} \right) \|u\|^2 + \frac{c}{2v} |u|^{2\beta} + \frac{cv}{2} |A_g u|^2 \\ &+ 2|u|(|f| + |h(t, u_t)|) \\ &\leq -2v \left(1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} \right) \|u\|^2 + \frac{cv\lambda_m}{2} \|u\|^2 + \delta \\ &\leq -2v \left(1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} - \frac{c\lambda_m}{4} \right) \|u\|^2 + \delta, \end{aligned}$$

where $\delta = \frac{c}{2v} (\tau k)^{2\beta} + 2\tau k(|f| + L_g \tau k)$.

For any $n > \tau$, as $|\nabla g|_\infty^2 < \frac{\lambda_1 m_0^2}{4}$, let $m = 1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} - \frac{c\lambda_m}{4}$, obviously $m > 0$, then $\frac{d}{dt} |u|^2 \leq -2vm \|u\|^2 + \delta$.

Multiply the above equation in t to $t+h$ ($t \geq t_2+h$), then we have

$$\begin{aligned} \int_t^{t+h} \frac{d}{dt} |u|^2 &\leq -2vm \int_t^{t+h} \|u\|^2 ds + \delta h, \\ 2vm \int_t^{t+h} \|u\|^2 ds &\leq |u|^2 + \delta h, \end{aligned}$$

so $\int_t^{t+h} \|u\|^2 ds \leq \frac{|u|^2 + \delta h}{2vm} \leq \frac{k^2}{2vm} + \frac{\delta h}{2vm}$. From (14) we obtain $\frac{d}{dt} \|u\|^2 + v(\lambda_1 - \frac{c\lambda_1}{2} - \frac{4|\nabla g|_\infty^2}{m_0^2}) \|u\|^2 \leq \frac{c|u|^{2\beta}}{2v} + \frac{2c'_1}{v^3} |u|^2 \|u\|^4 + \frac{8}{v} (|f|^2 + L_g^2 \|u\|_{L_{u_g}}^2)$, as $\lambda_1 - \frac{c\lambda_1}{2} - \frac{4|\nabla g|_\infty^2}{m_0^2} > 0$, that is $|\nabla g|_\infty^2 < \frac{\tilde{\lambda}_1 m_0^2}{4}$, where $\tilde{\lambda}_1 = \lambda_1 - \frac{c\lambda_1}{2}$. From the above formula, we obtain

$$\frac{d}{dt} \|u\|^2 \leq \frac{c|u|^{2\beta}}{2v} + \frac{2c'_1}{v^3} |u|^2 \|u\|^4 + \frac{8}{v} (|f|^2 + L_g^2 \|u\|_{L_{u_g}}^2).$$

According to the uniform Gronwall lemma, we have

$$\|u\|^2 \leq \left(\frac{a_3}{h} + a_2 \right) \exp(a_1), \quad \forall t \geq t_2 + T + h,$$

where $a_1 = \frac{2c'_1}{v^3} k^2 a_3$, $a_2 = \frac{8h}{v} (|f|^2 + L_g^2 k^2) + \frac{ch(\tau k)^{2\beta}}{2v}$, $a_3 = \frac{k^2}{2vm} + \frac{\delta h}{2vm}$.

Fixed h , then $B = B(0, \rho)$ is absorbed set in V_g . If B_0 is any bounded set in B , then $\forall t \geq t_2 + T + h$, we obtain $S(t)B_0 \subset B$.

Proof of Theorem 2 From Lemmas 1-3, we have a bounded absorbing set of (2) in C_{H_g} , and embedded $C_{V_g} \rightarrow C_{H_g}$ is compact. From the classical existence theo-

rem of attractors, a global attractor of the equation (2) is proved in C_{H_g} .

The proof of Theorem 3 is similar to that of Theorem 2. Here we omit the detailed proof.

Proof of Lemma 4 Using u_t to take inner product with $\frac{du}{dt} + vA_g u + Bu + G(u) + vRu = f + h(t, u_t)$, we have

$$\begin{aligned} \left(\frac{du}{dt}, u_t \right) + v(A_g u, u_t) + (Bu, u_t) + (G(u), u_t) + v(Ru, u_t) \\ = (f, u_t) + (h(t, u_t), u_t). \end{aligned}$$

Then $|u_t|^2 + \frac{v}{2} \frac{d}{dt} |\nabla u|^2 + (Bu, u_t) + (G(u), u_t) + v(Ru, u_t) = (f, u_t) + (h(t, u_t), u_t)$. So

$$\begin{aligned} |u_t|^2 + \frac{v}{2} \frac{d}{dt} |\nabla u|^2 + \frac{c}{\beta+1} \frac{d|u|^{\beta+1}}{dt} \\ \leq (f, u_t) + v(Ru, u_t) + (Bu, u_t) + (h(t, u_t), u_t) \\ \leq \frac{1}{4} |u_t|^2 + |f|^2 + \frac{1}{4} |u_t|^2 + c|u| \|u\|^2 |A_g u| \\ + \frac{v|\nabla g|_\infty}{m_0} \|u\| |u_t| + \frac{1}{4} |u_t|^2 + L_g^2 \|u(t)\|_{C_{u_g}}^2, \end{aligned}$$

where c is any constant. Therefore

$$\begin{aligned} \frac{1}{4} |u_t|^2 + \frac{v}{2} \frac{d}{dt} |\nabla u|^2 + \frac{c}{\beta+1} \frac{d|u|^{\beta+1}}{dt} \\ \leq |f|^2 + c|u| \|u\|^2 |A_g u| + \frac{v|\nabla g|_\infty}{2m_0} (\|u\|^2 + |u_t|^2) + L_g^2 \|u(t)\|_{C_{u_g}}^2. \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{2} \left(\frac{1}{2} - \frac{v|\nabla g|_\infty}{m_0} \right) |u_t|^2 + \frac{v}{2} \frac{d}{dt} |\nabla u|^2 + \frac{c}{\beta+1} \frac{d|u|^{\beta+1}}{dt} \\ \leq |f|^2 + c|u| \|u\|^2 |A_g u| + \frac{v|\nabla g|_\infty}{2m_0} \|u\|^2 + L_g^2 \|u(t)\|_{C_{u_g}}^2. \end{aligned}$$

Since $c > 0, \beta \geq 1$, as $|\nabla g|_\infty$ is sufficiently small, we make $\gamma = \frac{1}{2} - \frac{v|\nabla g|_\infty}{m_0} > 0$, that is $|\nabla g|_\infty < \frac{m_0}{2v}$, we get

$$\begin{aligned} \frac{v}{2} \frac{d}{dt} |\nabla u|^2 &\leq |f|^2 + c|u| \|u\|^2 |A_g u| + \frac{v|\nabla g|_\infty}{2m_0} \|u\|^2 + L_g^2 \|u(t)\|_{C_{u_g}}^2 \\ &\leq |f|^2 + \left(\frac{c}{4} + \frac{v|\nabla g|_\infty \lambda_1}{2m_0} \right) |A_g u|^2 + c|u|^2 \|u\|^4 + L_g^2 \|u(t)\|_{C_{u_g}}^2, \end{aligned} \tag{15}$$

where $\|u\|^2 \leq \lambda_1 |A_g u|^2, u \in D(A_g)$. Using $A_g u$ to take inner product with $\frac{du}{dt} + vA_g u + Bu + G(u) + vRu = f + h(t, u_t)$,

we obtain $\frac{1}{2} \frac{d}{dt} \|u\|^2 + v|A_g u|^2 + b(u, u, A_g u) + (G(u), A_g u) + v(Ru, A_g u) = (f, A_g u) + (h(t, u_t), A_g u)$. Since $(G(u), A_g u) \leq c \int |u|^{\beta-1} |u| |A_g u| dx \leq c|u|^{\beta-1} \|u\| \|A_g u\| \leq \frac{v}{8} |A_g u|^2 + c|u|^{2\beta}$.

Therefore

$$\begin{aligned} \frac{d}{dt} \|u\|^2 + 2v|A_g u|^2 \\ = 2(f, A_g u) - 2b(u, u, A_g u) - 2(G(u), A_g u) - 2v(Ru, A_g u) \\ + 2(h(t, u_t), A_g u) \end{aligned}$$

$$\begin{aligned} &\leq \frac{2}{\nu} |f|^2 + \frac{\nu}{2} |A_g u|^2 + \frac{\nu}{2} |A_g u|^2 + \frac{2c}{\nu} |u|^2 \|u\|^4 + \frac{\nu}{4} |A_g u|^2 \\ &\quad + 2c |u|^{2\beta} + \frac{2\nu |\nabla g|_\infty}{m_0} \|u\| |A_g u| + \frac{\nu}{2} |A_g u|^2 + \frac{2}{\nu} L_g^2 \|u(t)\|_{C_{H_g}}^2 \\ &\leq \frac{2}{\nu} |f|^2 + \frac{\nu}{2} |A_g u|^2 + \frac{7\nu}{2} |A_g u|^2 + 2c |u|^{2\beta} + \frac{2c}{\nu} |u|^2 \|u\|^4 \\ &\quad + \frac{2\nu \lambda_1^{1/2} |\nabla g|_\infty}{m_0} |A_g u|^2 + \frac{2}{\nu} L_g^2 \|u(t)\|_{C_{H_g}}^2. \\ &\text{So } \frac{d}{dt} \|u\|^2 + \nu \left(\frac{1}{4} - \frac{2\nu \lambda_1^{1/2} |\nabla g|_\infty}{m_0} \right) |A_g u|^2 \leq \frac{2}{\nu} |f|^2 + \frac{2c}{\nu} |u|^2 \\ &\|u\|^4 + 2c |u|^{2\beta} + \frac{2}{\nu} L_g^2 \|u(t)\|_{C_{H_g}}^2, \text{ then} \\ &\quad \nu \left(\frac{1}{4} - \frac{2\nu \lambda_1^{1/2} |\nabla g|_\infty}{m_0} \right) |A_g u|^2 \\ &\leq \frac{2}{\nu} |f|^2 + \frac{2c}{\nu} \rho_0^2 \rho_1^4 + 2c \rho_0^{2\beta} + \frac{2}{\nu} L_g^2 k^2. \end{aligned}$$

Integrating both sides of $[t, t + 1]$, when $|\nabla g|_\infty$ is sufficiently small, let $m = \frac{1}{4} - \frac{2\nu \lambda_1^{1/2} |\nabla g|_\infty}{m_0} > 0$, that is $|\nabla g|_\infty < \frac{m_0}{8\lambda_1^{1/2}}$, we have

$$\begin{aligned} m \int_t^{t+1} |A_g u|^2 ds &\leq \frac{2}{\nu} \int_t^{t+1} |f|^2 ds + \frac{2c}{\nu} \rho_0^2 \rho_1^4 + 2c \rho_0^{2\beta} + \frac{2}{\nu} L_g^2 k^2 \\ &= \frac{2}{\nu} |f|^2 + \frac{2c}{\nu} \rho_0^2 \rho_1^4 + 2c \rho_0^{2\beta} + \frac{2}{\nu} L_g^2 k^2. \end{aligned}$$

Then

$$\int_t^{t+1} |A_g u|^2 ds \leq \frac{1}{m} \left(\frac{2}{\nu} |f|^2 + \frac{2c}{\nu} \rho_0^2 \rho_1^4 + 2c \rho_0^{2\beta} + \frac{2}{\nu} L_g^2 k^2 \right).$$

From (15), we have $\frac{d}{dt} |\nabla u|^2 \leq \frac{2}{\nu} |f|^2 + \frac{2}{\nu} \left(\frac{c}{4} + \frac{\nu |\nabla g|_\infty \lambda_1}{2m_0} \right) |A_g u|^2 + \frac{2}{\nu} c \rho_0^2 \rho_1^4 + \frac{2}{\nu} L_g^2 k^2$.

Integrating both sides of $[t, t + 1]$, $|\nabla u(t + 1)|^2 \leq \frac{2}{\nu} |f|^2 + \frac{2}{\nu} \left(\frac{c}{4} + \frac{\nu |\nabla g|_\infty \lambda_1}{2m_0} \right) \int_t^{t+1} |A_g u|^2 ds + \frac{2}{\nu} c \rho_0^2 \rho_1^4 + \frac{2}{\nu} L_g^2 k^2 \leq 2|f|^2 + 2 \left(\frac{c}{4} + \frac{\nu |\nabla g|_\infty \lambda_1}{2m_0} \right) \frac{1}{m} \left(\frac{2}{\nu} |f|^2 + \frac{2c}{\nu} \rho_0^2 \rho_1^4 + 2c \rho_0^{2\beta} + \frac{2}{\nu} L_g^2 k^2 \right) + 2c \rho_0^2 \rho_1^4 + \frac{2}{\nu} L_g^2 k^2$. Taking a constant $M = 2|f|^2 + 2 \left(\frac{c}{4} + \frac{\nu |\nabla g|_\infty \lambda_1}{2m_0} \right) \frac{1}{m} \left(\frac{2}{\nu} |f|^2 + \frac{2c}{\nu} \rho_0^2 \rho_1^4 + 2c \rho_0^{2\beta} + \frac{2}{\nu} L_g^2 k^2 \right) + 2c \rho_0^2 \rho_1^4 + \frac{2}{\nu} L_g^2 k^2$, then $|\nabla(u + 1)|^2 \leq M$.

Proof of Theorem 4 Obviously $A_0 \subset A_1, \forall \zeta \in A_0$, from Theorem 3, for any $t_n = n$, we have $\{b_n\} \subset A_0$ and $S(n)b_n = \zeta, \forall t_1 > 0$, since $b_n \in A_0 = S(t_1)A_0$, so $\{c_n\} \subset A_0$, we have $S(t_1)c_n = b_n$. As A_0 is compact in C_{H_g} , by Lemma 4 and f is sufficiently small, we obtain that $\{S(t_1)c_n = b_n | t_1 \geq t_0\}$ is a bounded subset in C_{V_g} , therefore $\zeta = \lim_{n \rightarrow \infty} S(n)b_n \in A_1$. For ζ is arbitrary, we have $A_1 \subset A_0$, then

$$A_1 = A_0.$$

3 Conclusion

In this article, we study how to control the nonlinear dampness $c|u|^{\beta-1}u$ and time delay $h(t, u_t)$ to obtain the global attractor of the 2D gNSE on a bounded domain. We find that the global absorbing sets exist in C_{H_g} when $|\nabla g|_\infty > \frac{m_0 \lambda_1^{1/2}}{2} \left(1 + \frac{C_g}{\nu \lambda_1} \right)$, and exist in C_{V_g} when $0 < |\nabla g|_\infty < \frac{m_0 \lambda_0^{1/2}}{2} (\nu - C_g)$. We obtain the global attractor by the compact embedding method and find that the attractor has an asymptotic smoothing effect. The conclusions of this paper will further promote the research of NSE. It is necessary to study gNSE systematically. In the future, we may consider the global attractor 2D gNSE with damping and delay on the unbounded domain.

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含阻尼和时滞项的二维 g -Navier-Stokes 方程的全局吸引子

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摘要: 本文在空间 C_{H_g} 和空间 C_{V_g} 上分别得到了二维 g -Navier-Stokes 方程的全局吸引子。研究表明, 当外力项 f 充分小时, 在 C_{H_g} 和 C_{V_g} 上得到的全局吸引子是相等的。

关键词: 全局吸引子; g -Navier-Stokes 方程; 阻尼; 时滞

□