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New Proofs of Results about Proper Conflict-Free Coloring of Graphs

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Abstract: A proper conflict-free k-coloring of a graph is a proper k-coloring in which each nonisolated vertex has a color that appears exactly once in its open neighborhood. A graph is PCF k-colorable if it admits a proper conflict-free k- coloring. The PCF chromatic number of a graph G, denoted by $\chi_{pcf}(G)$, is the minimum k such that G is PCF k-colorable. Caro *et al* conjectured that for a connected graph G with maximum degree $\Delta \geqslant 3$, $\chi_{pcf}(G) \leqslant \Delta + 1$. One case in this conjecture, a connected graph with maximum degree 3 is PCF 4-colorable, can be derived from the result of Liu and Yu. Jiménez *et al* stated that the upper bound of PCF chromatic number of a graph G is $\max\{5, \chi(G)\}$ without a proof. In this paper, we give new proofs of the two results above and derive that for a connected graph G with maximum degree $\Delta \geqslant 3$, its complete subdivision is PCF $(\Delta+1)$-colorable.

Key words: proper conflict-free coloring; complete subdivision; minimal counterexample

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0 Introduction

All the graphs considered in this paper are finite, simple and undirected. We adopt the terminology and notation defined in Ref. [1] for any terms and symbols not explicitly defined here.

A proper conflict-free k-coloring (PCF k-coloring) of a graph is a proper k-coloring in which each nonisolated vertex has a unique color that appears exactly once in its open neighborhood. A graph is PCF k-colorable if it admits a proper conflict-free k-coloring. The minimum k such that G is PCF k-colorable is called the PCF chromatic number of G, denoted by $\chi_{pcf}(G)$. Let ϕ be a PCF coloring of a graph G. We denote the color on the vertex $v \in V(G)$ as $\phi(v)$ and its unique color as $\phi^*(v)$.

The concept of PCF coloring was introduced by Fabirici *et al*[2]. They proved that a planar graph has a PCF 8-coloring and constructed a planar graph with no PCF 5-coloring. This implies the upper bound of PCF chromatic number of planar graphs is between 6 and 8.

Caro *et al*[3] investigated PCF coloring further. They determined the PCF chromatic number for several well-studied graph classes, including trees, cycles, hypercubes and subdivision of complete graphs. In particular, they put forward the following conjecture.

Conjecture 1[3] If G is a connected graph of maximum degree $\Delta \geqslant 3$, then $\chi_{pcf}(G) \leqslant \Delta + 1$.

A linear coloring of a graph is a proper coloring where each pair of color classes induces a union of disjoint paths. Liu and Yu[4] showed the following theorem.

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Theorem 1 For every connected graph with maximum degree at most 3, there exists a linear 4-coloring such that the neighbors of every degree-two vertex receive different colors, unless the graph is C_5 or $K_{3,3}$.

By the definition of linear coloring and Theorem 1, a connected graph with maximum degree 3 is PCF 4-colorable, unless the graph is $K_{3,3}$. And it is not difficult to give $K_{3,3}$ a PCF 4-coloring. Thus a special case in Conjecture 1 can be derived easily and stated as follows.

Theorem 2 A connected graph with maximum degree 3 is PCF 4-colorable.

The complete subdivision of a graph G, denoted by $S(G)$, is a graph obtained from G by subdividing each edge in $E(G)$ exactly once. Caro *et al*[3] showed that $\chi_{\text{pcf}}(S(K_n))=n$ holds for $n\geqslant 3$. Additionally, they provided an upper bound on $\chi_{\text{pcf}}(S(G))$ when G has a 1-factor. In Ref. [5], the authors presented the following proposition without a proof.

Proposition 1[5] For any graph G, let $S(G)$ be the complete subdivision of G, then $\chi_{\text{pcf}}(S(G))\leqslant\max\{5,\chi(G)\}$.

Brook's Theorem states that $\chi(G)\leqslant\Delta+1$ for a connected graph G with maximum degree Δ, where equality holds if and only if G is a complete graph or an odd cycle. The following theorem follows from Theorem 2 and Proposition 1.

Theorem 3 If G is a connected graph with maximum degree $\Delta\geqslant 3$, then $\chi_{\text{pcf}}(S(G))\leqslant\Delta+1$.

More results about PCF coloring of graphs are in Refs. [6-17]. We give a new shorter proof of Theorem 2 in Section 1 and proofs of Proposition 1 and Theorem 3 in Section 2.

1 Proof of Theorem 2

Lemma 1 For a connected graph G on at least 6 vertices, if G has an edge $e\in E(G)$ such that $G-e$ has exactly one component isomorphic to C_5 and $G-V(C_5)$ is PCF 4-colorable, then G is also PCF 4-colorable.

Proof Let $e=uv_1$ be the edge such that $G-e$ has exactly one component isomorphic to C_5 and $V(C_5)=\{v_1,v_2,v_3,v_4,v_5\}$. And let ϕ be a PCF 4-coloring of $G-V(C_5)$. Assign v_1 a color not in $\{\phi(u),\phi^*(u)\}$. Next assign v_2 and v_5 the same color not in $\{\phi(v_1),\phi(u)\}$. Finally, assign v_3 and v_4 the remaining two colors not in $\{\phi(v_1),\phi(v_2)\}$. We obtain a PCF 4-coloring of G.

A set H of seven graphs is given in Fig. 1. For each graph in H, we give a PCF 4-coloring, i.e., they are all PCF 4-colorable. These graphs are useful in the following proof. In the following, we give the proof of Theorem 2.

Proof of Theorem 2 Let H be a counterexample with the minimum number of vertices. We shall complete the proof by several claims.

Claim 1 H has no 1-vertex.

Suppose H has a 1-vertex v. If $H-v\cong C_5$, then $H\cong H_1$. By Fig. 1, H is PCF 4-colorable. If $H-v\ncong C_5$, then by the choice of H, $H-v$ admits a PCF 4-coloring ϕ. Let u be the neighbor of v in H. By assigning a color other than $\phi(u)$ and $\phi^*(u)$ to v, we obtain a PCF 4-coloring of H. It is a contradiction.

Recall that an l-thread means a path on at least l vertices of degree 2 in a graph. Note that if a graph has no l-thread, then it has no $(l+1)$-thread.

Claim 2 There is no 3-thread in H.

Suppose H has a 3-thread $xuvwy$ where $d(u)=d(v)=d(w)=2$. By the choice of H, $H-\{u,v,w\}$ admits a PCF 4-coloring ϕ. Let

$$C=\{\phi(x),\phi^*(x),\phi(y),\phi^*(y)\}.$$

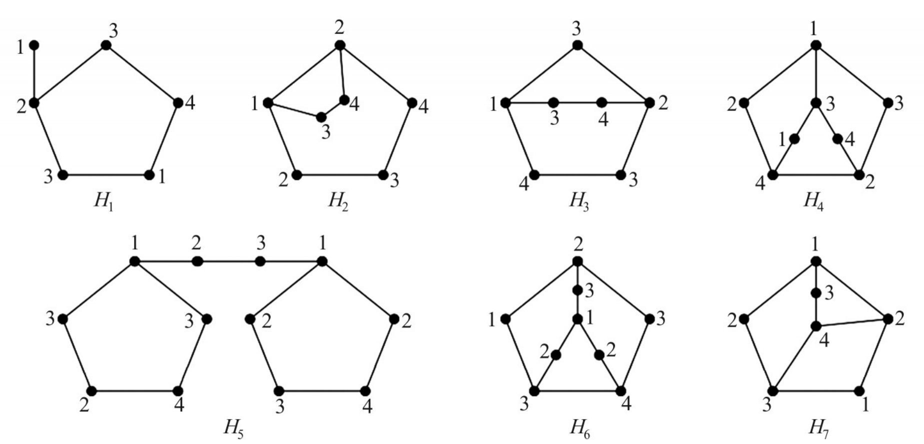

Fig. 1 $\boldsymbol{H=\{H_1,H_2,H_3,H_4,H_5,H_6,H_7\}}$

Note that $2 \leqslant |C| \leqslant 4$.

If $|C|=4$, color u, v, w by $\phi(y)$, $\phi^*(y)$, $\phi(x)$, respectively. We obtain a PCF 4-coloring of H.

If $|C|=3$, assign u a color in $\{\phi(y), \phi^*(y)\} \setminus \{\phi(x), \phi^*(x)\}$, v the remaining color and w a color in $\{\phi(x), \phi^*(x)\} \setminus \{\phi(y), \phi^*(y)\}$. We obtain a PCF 4-coloring of H.

If $|C|=2$, consider y. If $d(y)=2$, it is easy to replace $\phi(y)$ such that $|C|=3$. If $d(y)=3$, assign u a color not in $\{\phi(x), \phi^*(x)\}$, v a color not in $\{\phi(x), \phi(u), \phi(y)\}$ and w a color not in $\{\phi(u), \phi(v), \phi(y)\}$. We obtain a PCF 4-coloring of H.

Claim 3 Any two 2-vertices are not adjacent in H.

Suppose that there are two adjacent 2-vertices v and w in H. We consider two cases as follows.

Case 1 v and w have a common neighbor.

Let $N(v) \cap N(w)=\{z\}$. By Claim 2, $d(z)=3$. Let $N(z)=\{v, w, z_1\}$. By the choice of H, $H-v-w$ admits a PCF 4-coloring ϕ. Using two colors distinct from $\phi(z)$ and $\phi(z_1)$ to color v and w respectively, we obtain a PCF 4-coloring of H. It is a contradiction.

Case 2 v and w have no common neighbors.

Let $N(v) \setminus \{w\}=\{u\}$ and $N(w) \setminus \{v\}=\{x\}$. By Claim 2, $d(u)=d(x)=3$.

Consider $H-v-w$ in two subcases according to its connectivity.

Subcase 1 $H-v-w$ is connected.

If $H-v-w \cong C_5$, then $H \cong H_2$ or $H \cong H_3$. By Fig.1, H is PCF 4-colorable.

If $H-v-w \ncong C_5$, then by the choice of H, $H-v-w$ has a PCF 4-coloring ϕ. Assign w a color different from $\phi(x)$ and $\phi(u)$ and v a color not in $\{\phi(u), \phi(w), \phi(x)\}$. Thus, a PCF 4-coloring of H is obtained.

Subcase 2 $H-v-w$ is disconnected.

Clearly, $H-v-w$ has exactly two components. If both components are isomorphic to C_5, then $H \cong H_5$. Again, H is PCF 4-colorable.

If one component is C_5 and the other is not, then $H-V(C_5)$ is PCF 4-colorable.

By Lemma 1, H is PCF 4-colorable.

If no component of $H-v-w$ is C_5, then by the choice of H, $H-v-w$ has a PCF 4-coloring, which can be extended to a PCF 4-coloring of H in the same way in Subcase 1.

Claim 4 Each 3-vertex has at most two 2-neighbors in H.

By Claim 1, each vertex is of degree 2 or 3 in H. Suppose for the contrary that H has a 3-vertex u adjacent to three 2-vertices v, w and x. Let

$N(v)=\{u, v_1\}$, $N(w)=\{u, w_1\}$ and $N(x)=\{u, x_1\}$.

By Claim 3, we have $d(v_1)=d(w_1)=d(x_1)=3$. We consider $H-\{u, v, w, x\}$.

Case 3 $H-\{u, v, w, x\}$ is connected.

If $H-\{u, v, w, x\} \cong C_5$, then $H \cong H_6$. By Fig. 1, H is PCF 4-colorable.

If $H-\{u, v, w, x\} \ncong C_5$, then by the choice of H, $H-\{u, v, w, x\}$ admits a PCF 4-coloring ϕ. Assign u a color distinct from $\phi(v_1)$, $\phi(w_1)$ and $\phi(x_1)$. Next, assign v a color distinct from $\phi(u)$ and $\phi(v_1)$. Then assign w a color not in $\{\phi(u), \phi(v), \phi(w_1)$ and x a color not in $\{\phi(u), \phi(v), \phi(x_1)$. The resulting coloring is a PCF 4-coloring of H.

Case 4 $H-\{u, v, w, x\}$ is disconnected.

By Claim 3, a 5-cycle in H has at least three 3-vertices, which implies that no component of $H-\{u, v, w, x\}$ is C_5. Then by the choice of H, $H-\{u, v, w, x\}$ has a PCF 4-coloring, which can be extended to a PCF 4-coloring of H in the same way as we did in the previous case.

Claim 5 H has no 2-vertices.

Suppose that H has a 2-vertices u and $N(u)=\{v, w\}$. By Claim 1 and Claim 3, we know that $d(v)=d(w)=3$. Let $N(v)=\{u x, y\}$. By Claim 4, v has at most two 2-neighbors. We consider two cases as follows.

Case 5 v has just one 2-neighbors u.

Consider $H-u-v$.

Subcase 3 $H-u-v$ is connected.

If $H-u-v \cong C_5$, then $H \cong H_7$. By Fig.1, H is PCF 4-colorable.

If $H-u-v \ncong C_5$, then by the choice of H, $H-u-v$ admits a PCF 4-coloring ϕ. Assign v a color distinct from $\{\phi(x), \phi(y), \phi(w)\}$ and assgin u a color not in $\{\phi(w), \phi(v), \phi(x)\}$. The resulting coloring is a PCF 4-coloring of H.

Subcase 4 $H-u-v$ is disconnected.

Again, by Claim 3, no component of $H-u-v$ is C_5. Then by the choice of H, $H-u-v$ has a PCF 4-coloring, which can be extended to a PCF 4-coloring of H in the same way in the previous subcase.

Case 6 v has exactly two 2-neighbors.

With loss of generality, let $d(x)=2$, $d(y)=3$ and $N(x)=\{v, x_1\}$. By Claim 3, $d(x_1)=3$. Consider $H-\{u, v, x\}$.

Subcase 5 $H-\{u, v, x\}$ is connected.

By Claim 4, x_1, w and y have at most two 2-neighbors. If $H-\{u, v, x\}\cong C_5$, then $H\cong H_4$. Again H is PCF 4-colorable.

If $H-\{u, v, x\}\not\cong C_5$, then by the choice of H, $H-\{u, v, x\}$ admits a PCF 4-coloring ϕ. Assign v a color not in $\{\phi(w), \phi(y), \phi(x_1)\}$. Next, assign u a color not in $\{\phi(w), \phi(v), \phi(y)\}$ and assign x a color not in $\{\phi(x_1), \phi(v), \phi(y)$. Thus we obtain a PCF 4-coloring of H.

Subcase 6 $H-\{u, v, x\}$ is disconnected.

By Claim 3, no component of $H-\{u, v, x\}$ is C_5. Then by the choice of H, $H-\{u, v, x\}$ has a PCF 4-coloring, which can be extended to a PCF 4-coloring of H in the same way as we did in the previous subcase.

Next, we shall show that H is PCF 4-colorable to get a contradiction. By Claim 1 and Claim 5, H is 3-regular. We consider two cases as follows.

Claim 6 H has a triangle.

Assume that uvw is a triangle in H. Let $N(u)\backslash\{v, w\}=\{u_1\}$, $N(v)\backslash\{u, w\}=\{v_1\}$ and $N(w)\backslash\{u, v\}=\{w_1\}$. Note that $H-v$ has exactly three 2-vertices. Thus $H-v$ has no component isomorphic to C_5. By the choice of H, $H-v$ admits a PCF 4-coloring ϕ, which can be extended to a PCF 4-coloring of H after assign v a color not in $\{\phi(u), \phi(w), \phi(v_1)\}$. Thus H is PCF 4-colorable.

Claim 7 H is triangle-free.

Pick an edge $uv\in E(H)$, and let $N(u)\backslash\{v\}=\{u_1, u_2\}$ and $N(v)\backslash\{u\}=\{v_1, v_2\}$. Note that $H-u-v$ has exactly four 2-vertices. Thus $H-u-v$ has no component isomorphic to C_5. By the choice of H, $H-u-v$ admits a PCF 4-coloring ϕ.

If $\phi(v_1)=\phi(v_2)$, assign u a color not in $\{\phi(u_1), \phi(u_2), \phi(v_1)\}$, and then assign v a color not in $\{\phi(v_1), \phi(u), \phi(u_1)\}$. Thus, we can get a PCF 4-coloring of H.

If $\phi(v_1)\neq\phi(v_2)$, assign v a color not in $\{\phi(v_1), \phi(v_2), \phi(u_1)\}$, and then assign u a color not in $\{\phi(u_1), \phi(u_2), \phi(v)$. The resulting coloring is also a PCF 4-coloring of H.

This proof is completed.

2 Proofs of Proposition 1 and Theorem 3

Proof of Proposition 1 We may assume that G has no isolated vertex. Let M be a maximum matching of G. And let U be the set of unsaturated vertices of G under M. Thus, for each vertex $u\in U$, there exists a neighbor u' saturated by M in G. We now give a PCF coloring of $S(G)$ using at most $\max\{5, \chi(G)\}$ as follows.

Take a proper coloring ϕ of G using colors $\{1, \cdots, \chi(G)\}$. Color the vertex inserting into the edge uv of M with a color distinct from $\phi(u)$ and $\phi(v)$. This color is chosen to be the unique color (i.e. $\phi^*(u)$ and $\phi^*(v)$) on the neighborhoods of u and v. Next, for each $u\in U$, color the vertex inserting into uu' with a color distinct from $\phi(u)$, $\phi(u')$ and $\phi^*(u')$. This color is chosen to be the unique color (i.e. $\phi^*(u)$) on the neighborhoods of u. For an edge $xy\in E(G)\backslash(M\cup\{uu':u\in U\})$, color the vertex inserting into xy by a color not in $\{\phi(x), \phi^*(x), \phi(y), \phi^*(y)\}$. Thus, a PCF coloring of $S(G)$ is obtained using $\max\{5, \chi(G)\}$ colors.

Proof of Theorem 3 Let G be a connected graph with maximum degree $\Delta\geqslant 3$.

If $\Delta=3$, then $\Delta(S(G))=3$. By Theorem 2, $\chi_{\mathrm{pcf}}(S(G))=4=\Delta+1$.

If $\Delta\geqslant 4$, then by Brook's Theorem, $\chi(G)\leqslant\Delta+1$. By Proposition 1, $\chi_{\mathrm{pcf}}(S(G))\leqslant\max\{5, \chi(G)\}\leqslant\Delta+1$. The proof is completed.

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图的正常无冲突染色结论的新证明

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摘要：图的一个正常无冲突 k-染色是一个正常 k-染色，使得任意一个非孤立点的邻点中有一种颜色只出现一次。如果一个图有一个正常无冲突 k-染色，称它是正常无冲突 k-可染的。图的正常无冲突色数是使得它是正常无冲突 k-可染的 k 的最小值，记作 $\chi_{\mathrm{pcf}}(G)$。Caro 等人猜想 $\chi_{\mathrm{pcf}}(G) \leqslant \Delta+1$ 对最大度 $\Delta \geqslant 3$ 的连通图 G 成立。最大度为 3 的连通图是正常无冲突 4-可染的，是该猜想的一种情形，可由 Liu 和 Yu 的结论得到。Jiménez 等人不加证明地给出图 G 的正常无冲突色数的上界是 $\max\{5, \chi(G)\}$。本文中，我们给出上面两个结论新的证明，并得到对最大度 $\Delta \geqslant 3$ 的连通图，其完全剖分是正常无冲突 $(\Delta+1)$-可染的。

关键词：正常无冲突染色；完全剖分；极小反例法

□