Wuhan University Journal of Natural Sciences

2025, Vol.30 No.5, 458-462

Article ID 1007-1202(2025)05-0458-05 DOI https://doi.org/10.1051/wujns/2025305458

Cite this article: LI Pengfei. Existence of Entire Radial Solutions to Monge-Ampère Type Systems[J]. *Wuhan Univ J of Nat Sci*, 2025, 30(5): 458-462.

Existence of Entire Radial Solutions to Monge-Ampère Type Systems

□ **LI Pengfei**

School of Mathematics, Jilin University, Changchun 130012, Jilin, China

Abstract: This paper mainly studies the following Monge-Ampère type systems $\begin{cases}\det^{1/n}(\Delta u\boldsymbol{I}-D^2u)=p(|\boldsymbol{x}|)f(v), & \boldsymbol{x}\in\mathbb{R}^n,\\ \det^{1/n}(\Delta v\boldsymbol{I}-D^2v)=q(|\boldsymbol{x}|)g(u), & \boldsymbol{x}\in\mathbb{R}^n.\end{cases}$ The existence of entire radial solutions is obtained by using monotone iteration method and Arzelà-Ascoli theorem. These results generalize the classical Keller-Osserman condition to fully nonlinear systems.

Key words: Monge-Ampère type systems; entire radial solutions; Keller-Osserman condition

CLC number: O175.29

0 Introduction

In this paper, we study existence of entire convex radial solutions to the system

$$\begin{cases}\det^{1/n}(\Delta u\boldsymbol{I}-D^2u)=p(|\boldsymbol{x}|)f(v), & \boldsymbol{x}\in\mathbb{R}^n,\\ \det^{1/n}(\Delta v\boldsymbol{I}-D^2v)=q(|\boldsymbol{x}|)g(u), & \boldsymbol{x}\in\mathbb{R}^n.\end{cases} \tag{1}$$

Here $\mathbb{R}^n$ is n-dimensional Euclidean space and $n\geqslant 2$, D^2u denotes the Hessian matrix of u, $\Delta u=\sum_{i=1}^{n}\frac{\partial^2 u}{\partial \boldsymbol{x}_i{}^2}$ denotes the Laplacian of u, $\boldsymbol{I}$ is the identity matrix of order n, and we assume that p, q , f and g satisfy the following conditions:

(H_1) p, q: $[0,+\infty)\to(0,+\infty)$ are continuous;

(H_2) f, g: $[0,+\infty)\to[0,+\infty)$ are continuous and non-decreasing.

It is well known that for the following equation

$$\Delta u=f(u),\quad \boldsymbol{x}\in\mathbb{R}^n, \tag{2}$$

where f is a positive monotone increasing continuous function on $\mathbb{R}$, Keller[1] and Osserman[2] presented the famous Keller-Osserman condition:

$$\int^{\infty}\left(\int_0^t f(s)\ \mathrm{d}s\right)^{-\frac{1}{2}}\mathrm{d}t=\infty, \tag{3}$$

where we omit the lower limit to admit an arbitrary positive number. That is (2) has a sub-solution if and only if (3) holds. Ji and Bao[3] generalized the Laplace operator to the k-Hessian operator,

$$\sigma_k^{\frac{1}{k}}(\lambda(D^2u))=f(u),\quad \boldsymbol{x}\in\mathbb{R}^n.$$

Bao and Feng[4] generalized further to the p-k-Hessian equation. Bao *et al*[5] studied the Keller-Osserman type condition

Received date: 2024-10-15

Biography: LI Pengfei, male, Master candidate, research direction: nonlinear partial differential equation and geometric analysis. E-mail: lipf22@mails.jlu.edu.cn

for k-Yamabe type equations. Dai[6] investigated more general augmented Hessian equations of the following type and established generalized Keller-Osserman type condition,

$$\sigma_k^{\frac{1}{k}}(\lambda(D^2u+\alpha\boldsymbol{I}))=f(u),\quad \boldsymbol{x}\in\mathbb{R}^n.$$

Ji *et al*[7] studied a kind of k-Hessian equation with gradient terms and the following real $n-1$ Monge-Ampère equation in Ref. [8]:

$$\det{}^{1/n}(\Delta u\boldsymbol{I}-D^2u)=b(\boldsymbol{x})f(u),\quad \boldsymbol{x}\in\mathbb{R}^n. \tag{4}$$

The author considered three cases respectively: $b=1$, b is a spherically symmetric function, and b is a non-spherically symmetric function. The equation (4) originates from Gauduchon's conjecture[9] in complex geometry, which is a crucial conjecture in geometric analysis. Capuzzo Dolcetta *et al*[10] studied the Keller-Osserman type condition for degenerate second-order elliptic operators.

There is also a lot of work going to with system, for example,

$$\begin{cases}\Delta u=p(|\boldsymbol{x}|)v^{\alpha}, & \boldsymbol{x}\in\mathbb{R}^n,\\ \Delta v=q(|\boldsymbol{x}|)u^{\gamma}, & \boldsymbol{x}\in\mathbb{R}^n.\end{cases} \tag{5}$$

Here $n\geqslant 3$, $0<\alpha\leqslant\gamma$. Lair and Wood[11] studied the existence and nonexistence of entire positive radial solutions to the system. For the following system

$$\begin{cases}\det(D^2u)=f(-v), & \boldsymbol{x}\in B_1(0),\\ \det(D^2v)=f(-u), & \boldsymbol{x}\in B_1(0),\\ u=v=0, & \boldsymbol{x}\in\partial B_1(0),\end{cases} \tag{6}$$

Wang[12], Wang and An[13] studied the existence of convex radial solutions to (6), where $B_1(0)$ denotes the unit ball. Zhang and Zhou[14] considered the existence of entire positive radial solutions to the following system,

$$\begin{cases}\sigma_k(D^2u)=p(|\boldsymbol{x}|)f(v), & \boldsymbol{x}\in\mathbb{R}^n,\\ \sigma_k(D^2v)=q(|\boldsymbol{x}|)g(u), & \boldsymbol{x}\in\mathbb{R}^n.\end{cases}$$

Mi and Ji[15] extended the k-Hessian equations to the augmented p-k-Hessian systems and Cui[16] proved the existence and nonexistence of entire radial solutions to the k-Hessian type system with gradient terms. For more research on k-Hessian systems, refer to Refs. [17-20] and other relevant references.

We first introduce a fundamental lemma from Ref. [8].

Lemma 1 Let $r=|\boldsymbol{x}|=\left(\sum_{i=1}^{n}\boldsymbol{x}_i^{\,2}\right)^{1/2}$, $v(r)\in C^2[0,+\infty)$ be radially symmetric with $v'(0)=0$. Then for $u(\boldsymbol{x})=v(r)$, we have $u(\boldsymbol{x})\in C^2(\mathbb{R}^n)$ and

$$\det{}^{1/n}(\Delta u\boldsymbol{I}-D^2u)=\begin{cases}\dfrac{n-1}{r}v'(r)\left(v''(r)+\dfrac{n-2}{r}v'(r)\right)^{n-1}, & r\in(0,+\infty),\\ \left((n-1)v''(0)\right)^n, & r=0.\end{cases}$$

Moreover, $v(r)$ is a radial solution of (4) if and only if $v(r)$ satisfies the following ordinary differential equation

$$\frac{n-1}{r}v'(r)\left(v''(r)+\frac{n-2}{r}v'(r)\right)^{n-1}=b^n(r)f^n(v(r)),\quad r\in[0,+\infty).$$

This equation is equivalent to the following equation

$$v'(r)=\frac{r^{2-n}}{n-1}\left(\int_0^r ns^{n-1}p^{\frac{n}{n-1}}(s)f^{\frac{n}{n-1}}(v(s))\,\mathrm{d}s\right)^{\frac{n-1}{n}},\quad r\in[0,+\infty).$$

Subsequently, we introduce the main result of this paper. For system (1), we denote

$$P(\infty):=\lim_{r\to\infty}P(r),\ Q(\infty):=\lim_{r\to\infty}Q(r),\ H_a(\infty):=\lim_{r\to\infty}H_a(r);$$

$$P(r)=\int_0^r\left[\frac{t^{2-n}}{n-1}\left(\int_0^t ns^{n-1}p^{\frac{n}{n-1}}(s)\,\mathrm{d}s\right)^{\frac{n-1}{n}}\right]\mathrm{d}t,\ r\geqslant 0;\ Q(r)=\int_0^r\left[\frac{t^{2-n}}{n-1}\left(\int_0^t ns^{n-1}q^{\frac{n}{n-1}}(s)\,\mathrm{d}s\right)^{\frac{n-1}{n}}\right]\mathrm{d}t,\ r\geqslant 0;$$

$$H_a(r)=\int_a^r\frac{\mathrm{d}\tau}{f(\tau)+g(\tau)},\ r\geqslant a.$$

Theorem 1 Suppose (H_1) and (H_2) hold, $H_a(\infty)=\infty$, then (1) has one entire positive convex radial solution $u, v \in C^2(\mathbb{R}^n)$. Moreover,

(i) if $P(\infty)+Q(\infty)<\infty$, then u and v are bounded;

(ii) if $P(\infty)=Q(\infty)=\infty$, then $\lim\limits_{r\to\infty} u(r)=\infty$, $\lim\limits_{r\to\infty} v(r)=\infty$.

Theorem 2 Suppose (H_1) and (H_2) hold, $P(\infty)+Q(\infty)<H_a(\infty)=\infty$, then (1) has one entire positive convex radial solution $u, v \in C^2(\mathbb{R}^n)$. Moreover,

$$\frac{a}{2}+f\left(\frac{a}{2}\right)P(r)\leqslant u(r)\leqslant H_a^{-1}(P(r)+Q(r)),\ \forall r\geqslant 0;\ \frac{a}{2}+g\left(\frac{a}{2}\right)Q(r)\leqslant v(r)\leqslant H_a^{-1}(P(r)+Q(r)),\ \forall r\geqslant 0.$$

Remark 1 In a similar way, we can obtain the existence of entire radial solution to the following general system

$$\begin{cases}\det^{1/n}(\Delta u\boldsymbol{I}-D^2u)=p(|\boldsymbol{x}|)f_1(v)f_2(u), & \boldsymbol{x}\in\mathbb{R}^n,\\ \det^{1/n}(\Delta v\boldsymbol{I}-D^2v)=q(|\boldsymbol{x}|)g_1(v)g_2(u), & \boldsymbol{x}\in\mathbb{R}^n.\end{cases}\tag{7}$$

1 Proof of the Theorems

By Lemma 1, in order to find a radial solution of (1), we only need to prove the existence of solutions to the following system

$$\begin{cases}\dfrac{n-1}{r}u'(r)\left(u''(r)+\dfrac{n-2}{r}u'(r)\right)^{n-1}=p^n(r)f^n(v(r)), & r>0,\\ \dfrac{n-1}{r}v'(r)\left(v''(r)+\dfrac{n-2}{r}v'(r)\right)^{n-1}=q^n(r)g^n(u(r)), & r>0.\end{cases}$$

This is equivalent to the following system

$$\begin{cases}u(r)=\dfrac{a}{2}+\displaystyle\int_0^r\left[\frac{t^{2-n}}{n-1}\left(\int_0^t ns^{n-1}p^{\frac{n}{n-1}}(s)f^{\frac{n}{n-1}}(v(s))\,\mathrm{d}s\right)^{\frac{n-1}{n}}\right]\mathrm{d}t, & r\geqslant 0,\\ v(r)=\dfrac{a}{2}+\displaystyle\int_0^r\left[\frac{t^{2-n}}{n-1}\left(\int_0^t ns^{n-1}q^{\frac{n}{n-1}}(s)g^{\frac{n}{n-1}}(u(s))\,\mathrm{d}s\right)^{\frac{n-1}{n}}\right]\mathrm{d}t, & r\geqslant 0.\end{cases}$$

Let $\{u_m\}$ and $\{v_m\}$ be the sequences of positive continuous functions defined on $[0,+\infty)$ by

$$\begin{cases}u_0(r)=v_0(r)=\dfrac{a}{2},\\ u_m(r)=\dfrac{a}{2}+\displaystyle\int_0^r\left[\frac{t^{2-n}}{n-1}\left(\int_0^t ns^{n-1}p^{\frac{n}{n-1}}(s)f^{\frac{n}{n-1}}(v_{m-1}(s))\,\mathrm{d}s\right)^{\frac{n-1}{n}}\right]\mathrm{d}t, & r\geqslant 0,\\ v_m(r)=\dfrac{a}{2}+\displaystyle\int_0^r\left[\frac{t^{2-n}}{n-1}\left(\int_0^t ns^{n-1}q^{\frac{n}{n-1}}(s)g^{\frac{n}{n-1}}(u_{m-1}(s))\,\mathrm{d}s\right)^{\frac{n-1}{n}}\right]\mathrm{d}t, & r\geqslant 0.\end{cases}$$

By (H_1) and (H_2), we obtain $\forall r\geqslant 0$,

$$\frac{a}{2}\leqslant u_1(r)\leqslant u_2(r)\leqslant\cdots,\quad \frac{a}{2}\leqslant v_1(r)\leqslant v_2(r)\leqslant\cdots,$$

and

$$\begin{aligned}u_m'(r)&=\frac{r^{2-n}}{n-1}\left(\int_0^r ns^{n-1}p^{\frac{n}{n-1}}(s)f^{\frac{n}{n-1}}(v_{m-1}(s))\,\mathrm{d}s\right)^{\frac{n-1}{n}}\leqslant f(v_m(r))\frac{r^{2-n}}{n-1}\left(\int_0^r ns^{n-1}p^{\frac{n}{n-1}}(s)\mathrm{d}s\right)^{\frac{n-1}{n}}=f(v_m(r))P'(r)\\ &\leqslant\left[f(u_m(r)+v_m(r))+g(u_m(r)+v_m(r))\right]P'(r),\\ v_m'(r)&=\frac{r^{2-n}}{n-1}\left(\int_0^r ns^{n-1}q^{\frac{n}{n-1}}(s)g^{\frac{n}{n-1}}(u_{m-1}(s))\,\mathrm{d}s\right)^{\frac{n-1}{n}}\leqslant g(u_m(r))\frac{r^{2-n}}{n-1}\left(\int_0^r ns^{n-1}q^{\frac{n}{n-1}}(s)\mathrm{d}s\right)^{\frac{n-1}{n}}=g(u_m(r))Q'(r)\\ &\leqslant\left[f(u_m(r)+v_m(r))+g(u_m(r)+v_m(r))\right]Q'(r).\end{aligned}$$

Hence

$$u'_m(r)+v'_m(r)\leqslant \left[f(u_m(r)+v_m(r))+g(u_m(r)+v_m(r))\right]\left[P'(r)+Q'(r)\right].$$

We have

$$\int_a^{u_m(r)+v_m(r)}\frac{\mathrm{d}\tau}{f(\tau)+g(\tau)}\leqslant P(r)+Q(r), \qquad r>0,$$

$$H_a(u_m(r)+v_m(r))\leqslant P(r)+Q(r),$$

consequently,

$$u_m(r)+v_m(r)\leqslant H_a^{-1}(P(r)+Q(r)), \qquad \forall r\geqslant 0. \tag{8}$$

Hence the sequences $\{u_m\}$ and $\{v_m\}$ are bounded on $[0,R]$ for arbitrary $R>0$. $\forall m\in\mathbb{N},\ \forall x,y\in[0,R]$,

$$\begin{aligned}&|u_m(x)-u_m(y)|\\ &=\left|\int_0^x\left[\frac{t^{2-n}}{n-1}\left(\int_0^t ns^{n-1}p^{\frac{n}{n-1}}(s)f^{\frac{n}{n-1}}(v_{m-1}(s))\,\mathrm{d}s\right)^{\frac{n-1}{n}}\right]\mathrm{d}t-\int_0^y\left[\frac{t^{2-n}}{n-1}\left(\int_0^t ns^{n-1}p^{\frac{n}{n-1}}(s)f^{\frac{n}{n-1}}(v_{m-1}(s))\,\mathrm{d}s\right)^{\frac{n-1}{n}}\right]\mathrm{d}t\right|\\ &=\left|\int_x^y\left[\frac{t^{2-n}}{n-1}\left(\int_0^t ns^{n-1}p^{\frac{n}{n-1}}(s)f^{\frac{n}{n-1}}(v_{m-1}(s))\,\mathrm{d}s\right)^{\frac{n-1}{n}}\right]\mathrm{d}t\right|=\max_{r\in[0,R]}p(r)f(v_{m-1}(R))\left|\int_x^y\left[\frac{t^{2-n}}{n-1}\left(\int_0^t ns^{n-1}\,\mathrm{d}s\right)^{\frac{n-1}{n}}\right]\mathrm{d}t\right|\\ &=\max_{r\in[0,R]}p(r)f(v_{m-1}(R))\frac{1}{2(n-1)}|y^2-x^2|\leqslant\frac{R\max_{r\in[0,R]}p(r)f(v_{m-1}(R))}{n-1}|x-y|.\end{aligned}$$

Hence $\{u_m\}$ (and $\{v_m\}$) is equicontinuous on $[0,R]$ for arbitrary $R>0$.

By Arzelà-Ascoli theorem, $\{u_m\}$ and $\{v_m\}$ have subsequences $\{u_{m_k}\}$ and $\{v_{m_k}\}$ converge uniformly to u and v on $[0,R]$, respectively. Since $\{u_m\}$ and $\{v_m\}$ are monotonically increasing on $[0,\infty)$, $\{u_m\}$ and $\{v_m\}$ converge uniformly to u and v on $[0,R]$, respectively. By the arbitrariness of R, we obtain (u,v) is an entire positive k-convex radial solution of (1).

(i) $H_a(\infty)=\infty$.

If $P(\infty)+Q(\infty)<\infty$, by (8), we have $u(r)+v(r)\leqslant H_a^{-1}(P(\infty)+Q(\infty))$. So u and v are bounded.

If $P(\infty)=Q(\infty)=\infty$, then

$$u(r)\geqslant u_1(r)=\frac{a}{2}+f\left(\frac{a}{2}\right)\int_0^r\left[\frac{t^{2-n}}{n-1}\left(\int_0^t ns^{n-1}p^{\frac{n}{n-1}}(s)\,\mathrm{d}s\right)^{\frac{n-1}{n}}\right]\mathrm{d}t=\frac{a}{2}+f\left(\frac{a}{2}\right)P(r), \qquad \forall r\geqslant 0. \tag{9}$$

$$v(r)\geqslant v_1(r)=\frac{a}{2}+g\left(\frac{a}{2}\right)\int_0^r\left[\frac{t^{2-n}}{n-1}\left(\int_0^t ns^{n-1}q^{\frac{n}{n-1}}(s)\,\mathrm{d}s\right)^{\frac{n-1}{n}}\right]\mathrm{d}t=\frac{a}{2}+g\left(\frac{a}{2}\right)Q(r), \qquad \forall r\geqslant 0. \tag{10}$$

Let $r\to\infty$, we have $\lim_{r\to\infty}u(r)=\infty$, $\lim_{r\to\infty}v(r)=\infty$. Thus, the proof of Theorem 1 is completed.

(ii) $P(\infty)+Q(\infty)<H_a(\infty)=\infty$.

By (8), (9), and (10), we obtain

$$\frac{a}{2}+f\left(\frac{a}{2}\right)P(r)\leqslant u(r)\leqslant H_a^{-1}(P(r)+Q(r)), \forall r\geqslant 0;\ \frac{a}{2}+g\left(\frac{a}{2}\right)Q(r)\leqslant v(r)\leqslant H_a^{-1}(P(r)+Q(r)), \forall r\geqslant 0.$$

This is precisely what Theorem 2 aims to prove.

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Monge-Ampère型方程组整体镜像对称解的存在性

李鹏飞

吉林大学 数学学院，吉林 长春 130012

摘要：本文主要研究了下面的Monge-Ampère型方程组：$\begin{cases}\det^{1/n}(\Delta u\boldsymbol{I}-D^2u)=p(|\boldsymbol{x}|)f(v), & \boldsymbol{x}\in\mathbb{R}^n,\\ \det^{1/n}(\Delta v\boldsymbol{I}-D^2v)=q(|\boldsymbol{x}|)g(u), & \boldsymbol{x}\in\mathbb{R}^n.\end{cases}$ 利用单调迭代法和Arzelà-Ascoli定理，得到了整体镜像对称解的存在性。这些结果把经典的Keller-Osserman条件推广到了完全非线性方程组中。

关键词：Monge-Ampère型方程组；整体镜像对称解；Keller-Osserman条件

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