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Nonlinear Mixed Jordan Triple *-Higher Derivations on *-Algebras

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Abstract: Let $\mathcal{A}$ be a unital *-algebra containing a nontrivial projection, $\mathbb{N}$ be the set of non-negative integers. Under some mild conditions on $\mathcal{A}$, it is shown that any nonlinear mixed Jordan triple *-higher derivation $\mathcal{D}=\{d_n\}_{n\in\mathbb{N}}$ is an additive *-higher derivation. In particular, we apply the above result to prime *-algebras and von Neumann algebras with no central summands of type I_1.

Key words: mixed Jordan triple *-higher derivation; *-higher derivation; von Neumann algebra

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0 Introduction

Let $\mathcal{A}$ be a *-algebra over the complex field $\mathbb{C}$. For $A,B\in\mathcal{A}$, define the Jordan *-product of A and B by $A\bullet B=AB+BA^*$ and the skew Lie product of A and B by $[A,B]_*=AB-BA^*$. Among some research topics, the Jordan *-product and the skew Lie product are of great significance[1-9]. Recall that an additive map Φ: $\mathcal{A}\to\mathcal{A}$ is an additive derivation if $\Phi(AB)=\Phi(A)B+A\Phi(B)$ for all $A,B\in\mathcal{A}$. In addition, Φ is said to be an additive *-derivation if Φ is an additive derivation and $\Phi(A^*)=\Phi(A)^*$ for all $A\in\mathcal{A}$. Note that Φ is nonlinear if its additivity is removed.

Let $d:\mathcal{A}\to\mathcal{A}$ be a nonlinear map, d is called

(i) a nonlinear Jordan triple *-derivation if $d(A\bullet B\bullet C)=d(A)\bullet B\bullet C+A\bullet d(B)\bullet C+A\bullet B\bullet d(C)$ for all $A,B,C\in\mathcal{A}$, where $A\bullet B\bullet C=(A\bullet B)\bullet C$.

(ii) a nonlinear skew Lie triple derivation if $d([[A,B]_*,C]_*)=[[d(A),B]_*,C]_*+[[A,d(B)]_*,C]_*+[[A,B]_*,d(C)]_*$ for all $A,B,C\in\mathcal{A}$. The characterization of Jordan triple *-derivations and skew Lie triple derivations have been studied by many authors[10-19]. Recently the derivations corresponding to the new products of the mixture of various products have attracted the attentions of many researchers (see Refs. [20-22]). Li and Zhang[23] considered the mixed Jordan triple *-product $[A\bullet B,C]_*$ and investigated the nonlinear mixed Jordan triple *-derivation d: $\mathcal{A}\to\mathcal{A}$, i.e.

$$d([A\bullet B,C]_*)=[d(A)\bullet B,C]_*+[A\bullet d(B),C]_*+[A\bullet B,d(C)]_*,$$

and they proved a nonlinear mixed Jordan triple *-derivation is an additive *-derivation on *-algebras under some mild condition.

Let $\mathbb{N}$ be the set of non-negative integers, $\mathcal{D}=\{\Phi_n\}_{n\in\mathbb{N}}$ be a family of additive maps Φ_n: $\mathcal{A}\to\mathcal{A}$ and Φ_0 be the iden-

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tity map on $\mathcal{A}$. Then $\mathcal{D}=\{\Phi_n\}_{n\in\mathbb{N}}$ is called an additive higher derivation if for every $n\in\mathbb{N}$, $\Phi_n(AB)=\sum\limits_{i+j=n}\Phi_i(A)\Phi_j(B)$ for all $A,B\in\mathcal{A}$. $\mathcal{D}=\{\Phi_n\}_{n\in\mathbb{N}}$ is said to be an additive *-higher derivation if $\mathcal{D}=\{\Phi_n\}_{n\in\mathbb{N}}$ is an additive higher derivation with $\Phi_n(A^*)=\Phi_n(A)^*$ for all $A\in\mathcal{A}$ and $n\in\mathbb{N}$. In addition, $\mathcal{D}=\{\Phi_n\}_{n\in\mathbb{N}}$ is nonlinear if the additivity is removed. In recent years, many authors study different types of nonlinear higher derivations. Some researchers studied the nonlinear higher derivations on semi-prime rings and triangular algebras[24-26].

The questions of characterizing Lie higher derivations and the relationship between different types of higher derivations have been studied by many authors (see Refs. [27-29]).

Let d_0 be the identity map on $\mathcal{A}$, and let d_n: $\mathcal{A}\to\mathcal{A}$ be a nonlinear map for all $n\geqslant 1$. We say that $\mathcal{D}=\{d_n\}_{n\in\mathbb{N}}$ is

(i) a nonlinear Jordan triple *-higher derivation if $d_n(A\bullet B\bullet C)=\sum\limits_{i+j+k=n}d_i(A)\bullet d_j(B)\bullet d_k(C)$.

(ii) a nonlinear skew Lie triple higher derivation if $d_n([[A,B]_*,C]_*)=\sum\limits_{i+j+k=n}[[d_i(A),d_j(B)]_*,d_k(C)]$ for all $A,B,C\in\mathcal{A}$ and $n\in\mathbb{N}$. Some reasearchers proved that a nonlinear Jordan triple *-higher derivation or skew Lie triple higher derivation is an additive *-higher derivation on standard operator algebras[30-31]. Motivated by the related works[23, 30-31], we consider nonlinear mixed Jordan triple *-higher derivations $\mathcal{D}=\{d_n\}_{n\in\mathbb{N}}$, that is, $d_n([A\bullet B,C]_*)=\sum\limits_{i+j+k=n}[d_i(A)\bullet d_j(B),d_k(C)]_*$ for all $A,B,C\in\mathcal{A}$ and $n\in\mathbb{N}$. It is clear that d_1 is a nonlinear mixed Jordan triple *-derivation on $\mathcal{A}$ if $\mathcal{D}=\{d_n\}_{n\in\mathbb{N}}$ is a nonlinear mixed Jordan triple *-higher derivation of $\mathcal{A}$, then we will study the nonlinear mixed Jordan triple *-higher derivations on $\mathcal{A}$.

1 Nonlinear Mixed Jordan Triple *-Derivation

Theorem 1 Let $\mathcal{A}$ be a *-algebra with the unit I. Assume that $\mathcal{A}$ contains a nontrivial projection P which satisfies for $A\in\mathcal{A}$,

$$XAP=0 \text{ implies } X=0 \tag{1}$$

and

$$XA(I-P)=0 \text{ implies } X=0 \tag{2}$$

Then every nonlinear mixed Jordan triple *-derivation d:$\mathcal{A}\to\mathcal{A}$, which satisfies

$$d([A\bullet B,C]_*)=[d(A)\bullet B,C]_*+[A\bullet d(B),C]_*+[A\bullet B,d(C)]_*$$

for all $A,B,C\in\mathcal{A}$, is an additive *-derivation.

Proof Let $P_1=P$ and $P_2=I-P$. We write $\mathcal{A}_{jk}=P_j\mathcal{A}P_k$ for $j,\ k=1,2$. Then $\mathcal{A}=\sum\limits_{j,k=1}^{2}\mathcal{A}_{jk}$.

In the sequel $\mathcal{A}_{jk}$ indicates that $A_{jk}\in\mathcal{A}_{jk}$. We set $\mathcal{S}=\{A\in\mathcal{A},\ A=A^*\}$. We complete the proof by several Claims.

Claim 1 $d_n(0)=0$ for all $n\in\mathbb{N}$.

Proof It follows from $[0\bullet 0,0]_*=0$ that

$$d(0)=d([0\bullet 0,0]_*)=[d(0)\bullet 0,0]_*+[0\bullet d(0),0]_*+[0\bullet 0,d(0)]_*=0.$$

Claim 2 d has the following properties: (a) For any $\lambda\in\mathbb{R}$, $d(\lambda I)\in\mathcal{Z}(\mathcal{A})$ and $d(\lambda I)=d(\lambda I)^*$; (b) For any $A\in\mathcal{S}$, $d(A)=d(A^*)=d(A)^*$; (c) For any $\lambda\in\mathbb{C}$, $d(\lambda I)\in\mathcal{Z}(\mathcal{A})$.

Proof (a) It follows from $[I\bullet\lambda I,\ I]_*=0$ that

$$0=d\left([I\bullet\lambda I,I]_*\right)=[d(I)\bullet\lambda I,\ I]_*+[I\bullet d(\lambda I),\ I]_*+[I\bullet\lambda I,\ d(I)]_*=[I\bullet d(\lambda I),\ I]_*=2d(\lambda I)-2d(\lambda I)^*.$$

From this we get $d(\lambda I)=d(\lambda I)^*$. Then for all $A\in\mathcal{A}$, it follows from $[\lambda I\bullet\lambda I,\ A]_*=0$ that

$$\begin{aligned}0=d\left([\lambda I\bullet\lambda I,A]_*\right)&=[d(\lambda I)\bullet\lambda I,A]_*+[\lambda I\bullet d(\lambda I),A]_*+[\lambda I\bullet\lambda I,d(A)]_*\\&=[d(\lambda I)\bullet\lambda I,A]_*+[\lambda I\bullet d(\lambda I),A]_*=4\lambda d(\lambda I)A-4\lambda Ad(\lambda I)^*.\end{aligned}$$

Then we get $d(\lambda I)A=Ad(\lambda I)$, for all $A\in\mathcal{A}$, which implies $d(\lambda I)\in\mathcal{Z}(\mathcal{A})$.

(b) For any $A \in \mathcal{S}$, we see from Claim 2(a) that

$$0 = d\left(\left[I \bullet A, I\right]_*\right) = \left[d(I) \bullet A, I\right]_* + \left[I \bullet d(A), I\right]_* + \left[I \bullet A, d(I)\right]_* = \left[I \bullet d(A), I\right]_* = 2d(A) - 2d(A)^*.$$

(c) For any $\lambda \in \mathbb{C}$ and $A \in \mathcal{S}$, we see from Claim 2(a) and Claim 2(b) that

$$0 = d\left(\left[A \bullet I, \lambda I\right]_*\right) = \left[d(A) \bullet I, \lambda I\right]_* + \left[A \bullet d(I), \lambda I\right]_* + \left[A \bullet I, d(\lambda I)\right]_* = \left[A \bullet I, d(\lambda I)\right]_* = 2Ad(\lambda I) - 2d(\lambda I)A.$$

Thus $Ad(\lambda I) = d(\lambda I)A$, for all $A \in \mathcal{S}$, which implies $d(\lambda I) \in \mathcal{Z}(\mathcal{A})$.

Claim 3 For any $A \in \mathcal{A}$, $d\left(\frac{1}{2}I\right) = d\left(\frac{1}{2}\mathrm{i}I\right) = 0$, $d(\mathrm{i}A) = \mathrm{i}d(A)$.

Proof It follows from $\left[\frac{1}{2}I \bullet \frac{1}{2}\mathrm{i}I, \frac{1}{2}\mathrm{i}I\right]_* = -\frac{1}{2}I$, where i is the imaginary unit. We see from Claim 2 that

$$d\left(-\frac{1}{2}I\right) = d\left(\left[\frac{1}{2}I \bullet \frac{1}{2}\mathrm{i}I, \frac{1}{2}\mathrm{i}I\right]_*\right) = \left[d\left(\frac{1}{2}I\right) \bullet \frac{1}{2}\mathrm{i}I, \frac{1}{2}\mathrm{i}I\right]_* + \left[\frac{1}{2}I \bullet d\left(\frac{1}{2}\mathrm{i}I\right), \frac{1}{2}\mathrm{i}I\right]_* + \left[\frac{1}{2}I \bullet \frac{1}{2}\mathrm{i}I, d\left(\frac{1}{2}\mathrm{i}I\right)\right]_*$$

$$= -d\left(\frac{1}{2}I\right) + \frac{1}{2}\mathrm{i}\left(d\left(\frac{1}{2}\mathrm{i}I\right) - d\left(\frac{1}{2}\mathrm{i}I\right)^*\right) + \mathrm{i}d\left(\frac{1}{2}\mathrm{i}I\right).$$

Since $d(\frac{1}{2}I)$, $d\left(-\frac{1}{2}I\right)$, $\mathrm{i}\left(d\left(\frac{1}{2}\mathrm{i}I\right) - d\left(\frac{1}{2}\mathrm{i}I\right)^*\right)$ are self-adjoint, $\mathrm{i}d\left(\frac{1}{2}\mathrm{i}I\right)$ is also self-adjoint. Hence

$$d\left(-\frac{1}{2}I\right) = -d\left(\frac{1}{2}I\right) + 2\mathrm{i}d\left(\frac{1}{2}\mathrm{i}I\right). \tag{3}$$

Similarly, we can also obtain from the fact that $-\frac{1}{2}I = \left[\frac{1}{2}I \bullet \left(-\frac{1}{2}\mathrm{i}I\right), -\frac{1}{2}\mathrm{i}I\right]_*$, that

$$d\left(-\frac{1}{2}\mathrm{i}I\right) = -d\left(-\frac{1}{2}\mathrm{i}I\right)^* \tag{4}$$

and

$$d\left(-\frac{1}{2}I\right) = -d\left(\frac{1}{2}I\right) - 2\mathrm{i}d\left(-\frac{1}{2}\mathrm{i}I\right). \tag{5}$$

Now by (3) and (5) we have

$$d\left(-\frac{1}{2}\mathrm{i}I\right) = -d\left(\frac{1}{2}\mathrm{i}I\right). \tag{6}$$

It follows from (4) and (6), we compute

$$d\left(\frac{1}{2}\mathrm{i}I\right) = d\left(\left[\frac{1}{2}I \bullet \left(-\frac{1}{2}\mathrm{i}I\right), -\frac{1}{2}I\right]_*\right)$$

$$= \left[d\left(\frac{1}{2}I\right) \bullet \left(-\frac{1}{2}\mathrm{i}I\right), -\frac{1}{2}I\right]_* + \left[\frac{1}{2}I \bullet d\left(-\frac{1}{2}\mathrm{i}I\right), -\frac{1}{2}I\right]_* + \left[\frac{1}{2}I \bullet \left(-\frac{1}{2}\mathrm{i}I\right), d\left(-\frac{1}{2}I\right)\right]_*$$

$$= \mathrm{i}d\left(\frac{1}{2}I\right) - \frac{1}{2}\left(d\left(-\frac{1}{2}\mathrm{i}I\right) - d\left(-\frac{1}{2}\mathrm{i}I\right)^*\right) - \mathrm{i}d\left(-\frac{1}{2}I\right).$$

Then we can get

$$d\left(\frac{1}{2}I\right) = d\left(-\frac{1}{2}I\right). \tag{7}$$

Now we compute

$$d(-\frac{1}{2}\mathrm{i}I) = d\left(\left[-\frac{1}{2}I \bullet \left(-\frac{1}{2}\mathrm{i}I\right), -\frac{1}{2}I\right]_*\right)$$

$$= \left[d\left(-\frac{1}{2}I\right) \bullet \left(-\frac{1}{2}\mathrm{i}I\right), -\frac{1}{2}I\right]_* + \left[-\frac{1}{2}I \bullet d\left(-\frac{1}{2}\mathrm{i}I\right), -\frac{1}{2}I\right]_* + \left[-\frac{1}{2}I \bullet \left(-\frac{1}{2}\mathrm{i}I\right), d\left(-\frac{1}{2}I\right)\right]_*$$

$$= 2\mathrm{i}d\left(-\frac{1}{2}I\right) + \frac{1}{2}\left(d\left(-\frac{1}{2}\mathrm{i}I\right) - d\left(-\frac{1}{2}\mathrm{i}I\right)^*\right).$$

By using equation (4) we get $d\left(-\frac{1}{2}I\right)=0$, then by equations (3), (6) and (7), we have $d\left(-\frac{1}{2}I\right)=d\left(\frac{1}{2}I\right)=d\left(-\frac{1}{2}\mathrm{i}I\right)=d\left(\frac{1}{2}\mathrm{i}I\right)=0$.

For any $A\in\mathcal{A}$ we have

$$d(\mathrm{i}A)=d\left(\left[\frac{1}{2}I\bullet\frac{1}{2}\mathrm{i}I,\ A\right]_*\right)=\left[d\left(\frac{1}{2}I\right)\bullet\frac{1}{2}\mathrm{i}I,\ A\right]_*+\left[\frac{1}{2}I\bullet d\left(\frac{1}{2}\mathrm{i}I\right),\ A\right]_*+\left[\frac{1}{2}I\bullet\frac{1}{2}\mathrm{i}I,\ d(A)\right]_*=\left[\frac{1}{2}I\bullet\frac{1}{2}\mathrm{i}I,\ d(A)\right]_*=\mathrm{i}d(A).$$

Claim 4[23] d is additive.

Claim 5 For any $A\in\mathcal{A}$, $d(A^*)=d(A)^*$.

Proof For any $A\in\mathcal{A}$, where $A=A_1+\mathrm{i}A_2$, $A_1,A_2\in\mathcal{S}$, it follows from Claim 2, Claim 3 and Claim 4 that

$$d(A^*)=d(A_1-\mathrm{i}A_2)=d(A_1)-\mathrm{i}d(A_2)=d(A_1)^*+\left(\mathrm{i}d(A_2)\right)^*=d(A_1+\mathrm{i}A_2)^*=d(A)^*.$$

Claim 6 d is an additive *-derivation on $\mathcal{A}$.

Proof To complete the proof, we only need to show that d is a derivation on $\mathcal{A}$. Since d is additive. $[A,B]_*=AB-BA^*$ and $[\mathrm{i}A,\mathrm{i}B]_*=-AB-BA^*$ for all $A,B\in\mathcal{A}$. It follows from Claim 3 that

$$\begin{aligned}d(AB)-d(BA^*)=d([A,B]_*)&=d\left(\left[\frac{1}{2}I\bullet A,\ B\right]_*\right)=\left[d\left(\frac{1}{2}I\right)\bullet A,\ B\right]_*+\left[\frac{1}{2}I\bullet d(A),\ B\right]_*+\left[\frac{1}{2}I\bullet A,\ d(B)\right]_*\\&=\left[\frac{1}{2}I\bullet d(A),\ B\right]_*+\left[\frac{1}{2}I\bullet A,\ d(B)\right]_*=d(A)B-Bd(A)^*+Ad(B)-d(B)A^*.\end{aligned}$$

And

$$\begin{aligned}-d(AB)-d(BA^*)=d([\mathrm{i}A,\mathrm{i}B]_*)&=d\left(\left[\frac{1}{2}I\bullet \mathrm{i}A,\mathrm{i}B\right]_*\right)=\left[d\left(\frac{1}{2}I\right)\bullet \mathrm{i}A,\mathrm{i}B\right]_*+\left[\frac{1}{2}I\bullet d(\mathrm{i}A),\mathrm{i}B\right]_*+\left[\frac{1}{2}I\bullet \mathrm{i}A,d(\mathrm{i}B)\right]_*\\&=\left[\frac{1}{2}I\bullet d(\mathrm{i}A),\mathrm{i}B\right]_*+\left[\frac{1}{2}I\bullet \mathrm{i}A,\ d(\mathrm{i}B)\right]_*=-d(A)B-Bd(A)^*-Ad(B)-d(B)A^*.\end{aligned}$$

Using the two equations above, we obtain $d(AB)=d(A)B+Ad(B)$, and Claim 6 is proved.

It follows from Claims 4-6 that d is an additive *-derivation on $\mathcal{A}$, which completes the proof of Theorem 1.

2 Nonlinear Mixed Jordan Triple *-Higher Derivation

Theorem 2 Let $\mathcal{A}$ be a *-algebra with the unit I, and let $\mathbb{N}$ be the set of non-negative integers. Assume that $\mathcal{A}$ contains a nontrivial projection P which satisfies for $A\in\mathcal{A}$, $XAP=0$ implies $X=0$, and $XA(I-P)=0$ implies $X=0$.

Then every nonlinear mixed Jordan triple *-higher derivation $\mathcal{D}=\{d_n\}_{n\in\mathbb{N}}$ on $\mathcal{A}$ which satisfies

$$d_n([A\bullet B,C]_*)=\sum_{i+j+k=n}[d_i(A)\bullet d_j(B),\ d_k(C)]_*$$

for all $A,\ B,\ C\in\mathcal{A}$ and $n\in\mathbb{N}$, is an additive *-higher derivation.

Proof Let $P_1=P$ and $P_2=I-P$. We write $\mathcal{A}_{jk}=P_j\mathcal{A}P_k$ for $j,\ k=1,2$. Then $\mathcal{A}=\sum_{j,k=1}^{2}\mathcal{A}_{jk}$.

In the sequel $\mathcal{A}_{jk}$ indicates that $A_{jk}\in\mathcal{A}_{jk}$. We set $\mathcal{S}=\{A\in\mathcal{A},\ A=A^*\}$. We complete the proof by several Claims.

Claim 7 $d_n(0)=0$ for all $n\in\mathbb{N}$.

Proof It follows from Claim 1 that $d_1(0)=0$. Now assume that $d_k(0)=0$ for $k<n$. Then we compute

$$d_n(0)=d_n\left([0\bullet 0,0]_*\right)=\sum_{m+l+j=n}\left[d_m(0)\bullet d_l(0),d_j(0)\right]_*=\sum_{\substack{m+l+j=n\\ j<n}}\left[d_m(0)\bullet d_l(0),d_j(0)\right]_*+\left[d_0(0)\bullet d_0(0),d_n(0)\right]_*=0.$$

Claim 8 For any $n\in\mathbb{N}$, d_n has the following properties: (a) For any $\lambda\in\mathbb{R}$, $d_n(\lambda I)\in\mathcal{Z}(\mathcal{A})$ and $d_n(\lambda I)=d_n(\lambda I)^*$; (b) For any $A\in\mathcal{S}$, $d_n(A)=d_n(A^*)=d_n(A)^*$; (c) For any $\lambda\in\mathbb{C}$, $d_n(\lambda I)\in\mathcal{Z}(\mathcal{A})$.

Proof It follows from Claim 2 and Claim 6 that $d_1(\lambda I)=0$ for $\lambda\in\mathbb{C}$ and $d_1(A^*)=d_1(A)^*$ for $A\in\mathcal{A}$.

(a) For any $\lambda\in\mathbb{R}$, we assume that $d_k(\lambda I)=d_k(\lambda I)^*\in\mathcal{Z}(\mathcal{A})$ for $k<n$. Then

$$0=d_n\left([I\bullet\lambda I, I]_*\right)=\sum_{m+l+j=n}[d_m(I)\bullet d_l(\lambda I), d_j(I)]_*$$

$$=[d_n(I)\bullet\lambda I, I]_*+[I\bullet d_n(\lambda I), I]_*+[I\bullet\lambda I, d_n(I)]_*+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}[d_m(I)\bullet d_l(\lambda I), d_j(I)]_*$$

$$=[d_n(I)\bullet\lambda I, I]_*+[I\bullet d_n(\lambda I), I]_*+[I\bullet\lambda I, d_n(I)]_*=2d_n(\lambda I)-2d_n(\lambda I)^*.$$

From this we get $d_n(\lambda I)=d_n(\lambda I)^*$. Then for all $A\in\mathcal{A}$,

$$0=d_n\left([\lambda I\bullet\lambda I, A]_*\right)=\sum_{m+l+j=n}[d_m(\lambda I)\bullet d_l(\lambda I), d_j(A)]_*$$

$$=[d_n(\lambda I)\bullet\lambda I, A]_*+[\lambda I\bullet d_n(\lambda I), A]_*+[\lambda I\bullet\lambda I, d_n(A)]_*+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}[d_m(\lambda I)\bullet d_1(\lambda I), d_j(A)]_*$$

$$=[d_n(\lambda I)\bullet\lambda I, A]_*+[\lambda I\bullet d_n(\lambda I), A]_*+[\lambda I\bullet\lambda I, d_n(A)]_*=4\lambda d_n(\lambda I)A-4\lambda Ad_n(\lambda I)^*.$$

So we get $d_n(\lambda I)A=Ad_n(\lambda I)$ for all $A\in\mathcal{A}$, which implies $d_n(\lambda I)\in\mathcal{Z}(\mathcal{A})$.

(b) For any $A\in\mathcal{S}$, we assume that $d_k(A)=d_k(A^*)=d_k(A)^*$ for $k<n$. Then we can get from Claim 8(a) that

$$0=d_n\left([I\bullet A, I]_*\right)=\sum_{m+l+j=n}[d_m(I)\bullet d_l(A), d_j(I)]_*$$

$$=[d_n(I)\bullet A, I]_*+[I\bullet d_n(A), I]_*+[I\bullet A, d_n(I)]_*+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}[d_m(I)\bullet d_l(A), d_j(I)]_*$$

$$=[d_n(I)\bullet A, I]_*+[I\bullet d_n(A), I]_*+[I\bullet A, d_n(I)]_*=2d_n(A)-2d_n(A)^*.$$

(c) For any $\lambda\in\mathbb{C}$ and $A\in\mathcal{S}$, assume that $d_k(\lambda I)\in\mathcal{Z}(\mathcal{A})$ for $k<n$. Then we see from Claim 8(a) and 8(b) that

$$0=d_n\left([A\bullet I, \lambda I]_*\right)=\sum_{m+l+j=n}[d_m(A)\bullet d_l(I), d_j(\lambda I)]_*$$

$$=[d_n(A)\bullet I, \lambda I]_*+[A\bullet d_n(I), \lambda I]_*+[A\bullet I, d_n(\lambda I)]_*+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}[d_m(A)\bullet d_l(I), d_j(\lambda I)]_*$$

$$=[d_n(A)\bullet I, \lambda I]_*+[A\bullet d_n(I), \lambda I]_*+[A\bullet I, d_n(\lambda I)]_*=2Ad_n(\lambda I)-2d_n(\lambda I)A.$$

Thus $Ad_n(\lambda I)=d_n(\lambda I)A$ for all $A\in\mathcal{S}$. This implies that $d_n(\lambda I)\in\mathcal{Z}(\mathcal{A})$.

Claim 9 For any $A\in\mathcal{A}$ and $n\in\mathbb{N}^+$, $d_n\left(\frac{1}{2}I\right)=d_n\left(\frac{1}{2}\mathrm{i}I\right)=0,\ d_n(\mathrm{i}A)=\mathrm{i}d_n(A)$.

Proof It follows from Claim 3 that $d_1\left(\frac{1}{2}I\right)=d_1\left(\frac{1}{2}\mathrm{i}I\right)=0,\ d_1(\mathrm{i}A)=\mathrm{i}d_1(A)$. Now assume $d_k\left(\frac{1}{2}I\right)=d_k\left(\frac{1}{2}\mathrm{i}I\right)=0,\ d_k(\mathrm{i}A)=\mathrm{i}d_k(A)$ for $k<n$. Then by Claim 8,

$$d_n\left(-\frac{1}{2}I\right)=d_n\left(\left[\frac{1}{2}I\bullet\frac{1}{2}\mathrm{i}I, \frac{1}{2}\mathrm{i}I\right]_*\right)=\sum_{m+l+j=n}\left[d_m\left(\frac{1}{2}I\right)\bullet d_l\left(\frac{1}{2}\mathrm{i}I\right), d_j\left(\frac{1}{2}\mathrm{i}I\right)\right]_*$$

$$=\left[d_n\left(\frac{1}{2}I\right)\bullet\frac{1}{2}\mathrm{i}I, \frac{1}{2}\mathrm{i}I\right]_*+\left[\frac{1}{2}I\bullet d_n\left(\frac{1}{2}\mathrm{i}I\right), \frac{1}{2}\mathrm{i}I\right]_*+\left[\frac{1}{2}I\bullet\frac{1}{2}\mathrm{i}I, d_n\left(\frac{1}{2}\mathrm{i}I\right)\right]_*+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}\left[d_m\left(\frac{1}{2}I\right)\bullet d_l\left(\frac{1}{2}\mathrm{i}I\right), d_j\left(\frac{1}{2}\mathrm{i}I\right)\right]_*$$

$$=\left[d_n\left(\frac{1}{2}I\right)\bullet\frac{1}{2}\mathrm{i}I, \frac{1}{2}\mathrm{i}I\right]_*+\left[\frac{1}{2}I\bullet d_n\left(\frac{1}{2}\mathrm{i}I\right), \frac{1}{2}\mathrm{i}I\right]_*+\left[\frac{1}{2}I\bullet\frac{1}{2}\mathrm{i}I, d_n\left(\frac{1}{2}\mathrm{i}I\right)\right]_*$$

$$=-d_n\left(\frac{1}{2}I\right)+\frac{1}{2}\mathrm{i}\left(d_n\left(\frac{1}{2}\mathrm{i}I\right)-d_n\left(\frac{1}{2}\mathrm{i}I\right)^*\right)+\mathrm{i}d_n\left(\frac{1}{2}\mathrm{i}I\right).$$

Since $d_n(\frac{1}{2}I)$, $d_n\left(-\frac{1}{2}I\right)$, $\mathrm{i}(d_n\left(\frac{1}{2}\mathrm{i}I\right)-d_n\left(\frac{1}{2}\mathrm{i}I\right)^*)$ are self-adjoint, $\mathrm{i}d_n\left(\frac{1}{2}\mathrm{i}I\right)$ is also self-adjoint. Hence

$$d_n\left(-\frac{1}{2}I\right)=-d_n\left(\frac{1}{2}I\right)+2\mathrm{i}d_n\left(\frac{1}{2}\mathrm{i}I\right). \tag{8}$$

Similarly, we can obtain from the fact that $-\frac{1}{2}I=\left[\frac{1}{2}I\bullet\left(-\frac{1}{2}\mathrm{i}I\right),-\frac{1}{2}\mathrm{i}I\right]_*$, then

$$d_n\left(-\frac{1}{2}\mathrm{i}I\right)=-d_n\left(-\frac{1}{2}\mathrm{i}I\right)^* \tag{9}$$

and

$$d_n\left(-\frac{1}{2}I\right)=-d_n\left(\frac{1}{2}I\right)-2\mathrm{i}d_n\left(-\frac{1}{2}\mathrm{i}I\right). \tag{10}$$

Now by (8) and (10) we have

$$d_n\left(-\frac{1}{2}\mathrm{i}I\right)=-d_n\left(\frac{1}{2}\mathrm{i}I\right). \tag{11}$$

Then we compute

$$\begin{aligned}
d_n\left(\frac{1}{2}\mathrm{i}I\right)&=d_n\left(\left[\frac{1}{2}I\bullet\left(-\frac{1}{2}\mathrm{i}I\right),-\frac{1}{2}I\right]_*\right)=\sum_{m+l+j=n}\left[d_m\left(\frac{1}{2}I\right)\bullet d_l\left(-\frac{1}{2}\mathrm{i}I\right),d_j\left(-\frac{1}{2}I\right)\right]_*\\
&=\left[d_n\left(\frac{1}{2}I\right)\bullet\left(-\frac{1}{2}\mathrm{i}I\right),-\frac{1}{2}I\right]_*+\left[\frac{1}{2}I\bullet d_n\left(-\frac{1}{2}\mathrm{i}I\right),-\frac{1}{2}I\right]_*+\left[\frac{1}{2}I\bullet\left(-\frac{1}{2}\mathrm{i}I\right),d_n\left(-\frac{1}{2}I\right)\right]_*\\
&\quad+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}\left[d_m\left(\frac{1}{2}I\right)\bullet d_l\left(-\frac{1}{2}\mathrm{i}I\right),d_j\left(-\frac{1}{2}I\right)\right]_*\\
&=\left[d_n\left(\frac{1}{2}I\right)\bullet\left(-\frac{1}{2}\mathrm{i}I\right),-\frac{1}{2}I\right]_*+\left[\frac{1}{2}I\bullet d_n\left(-\frac{1}{2}\mathrm{i}I\right),-\frac{1}{2}I\right]_*+\left[\frac{1}{2}I\bullet\left(-\frac{1}{2}\mathrm{i}I\right),d_n\left(-\frac{1}{2}I\right)\right]_*\\
&=\mathrm{i}d_n\left(\frac{1}{2}I\right)-\frac{1}{2}I\left(d_n\left(-\frac{1}{2}\mathrm{i}I\right)-d_n\left(-\frac{1}{2}\mathrm{i}I\right)^*\right)-\mathrm{i}d_n\left(-\frac{1}{2}I\right).
\end{aligned}$$

By using (9) and (11) we get

$$d_n\left(\frac{1}{2}I\right)=d_n\left(-\frac{1}{2}I\right). \tag{12}$$

Now compute

$$\begin{aligned}
d_n\left(-\frac{1}{2}\mathrm{i}I\right)&=d_n\left(\left[-\frac{1}{2}I\bullet\left(-\frac{1}{2}\mathrm{i}I\right),-\frac{1}{2}I\right]_*\right)=\sum_{m+l+j=n}\left[d_m\left(-\frac{1}{2}I\right)\bullet d_l\left(-\frac{1}{2}\mathrm{i}I\right),d_j\left(-\frac{1}{2}I\right)\right]_*\\
&=\left[d_n\left(-\frac{1}{2}I\right)\bullet\left(-\frac{1}{2}\mathrm{i}I\right),-\frac{1}{2}I\right]_*+\left[-\frac{1}{2}I\bullet d_n\left(-\frac{1}{2}\mathrm{i}I\right),-\frac{1}{2}I\right]_*\\
&\quad+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}\left[d_m\left(-\frac{1}{2}I\right)\bullet d_l\left(-\frac{1}{2}\mathrm{i}I\right),d_j\left(-\frac{1}{2}I\right)\right]_*+\left[-\frac{1}{2}I\bullet\left(-\frac{1}{2}\mathrm{i}I\right),d_n\left(-\frac{1}{2}I\right)\right]_*\\
&=\left[d_n\left(-\frac{1}{2}I\right)\bullet\left(-\frac{1}{2}\mathrm{i}I\right),-\frac{1}{2}I\right]_*+\left[-\frac{1}{2}I\bullet d_n\left(-\frac{1}{2}\mathrm{i}I\right),-\frac{1}{2}I\right]_*+\left[-\frac{1}{2}I\bullet\left(-\frac{1}{2}\mathrm{i}I\right),d_n\left(-\frac{1}{2}I\right)\right]_*\\
&=2\mathrm{i}d_n\left(-\frac{1}{2}I\right)+\frac{1}{2}\left(d_n\left(-\frac{1}{2}\mathrm{i}I\right)-d_n\left(-\frac{1}{2}\mathrm{i}I\right)^*\right).
\end{aligned}$$

By using (9) we get $d_n\left(-\frac{1}{2}I\right)=0$, then by equations (8), (11) and (12), we have $d_n\left(-\frac{1}{2}I\right)=d_n\left(\frac{1}{2}I\right)=d_n\left(-\frac{1}{2}\mathrm{i}I\right)=d_n\left(\frac{1}{2}\mathrm{i}I\right)=0$.

For any $A\in\mathcal{A}$ we have

$$\begin{aligned}
d_n(\mathrm{i}A)&=d_n\left(\left[\frac{1}{2}I\bullet\frac{1}{2}\mathrm{i}I,A\right]_*\right)=\sum_{m+l+j=n}\left[d_m\left(\frac{1}{2}I\right)\bullet d_l\left(\frac{1}{2}\mathrm{i}I\right),d_j(A)\right]_*\\
&=\left[d_n\left(\frac{1}{2}I\right)\bullet\frac{1}{2}\mathrm{i}I,A\right]_*+\left[\frac{1}{2}I\bullet d_n\left(\frac{1}{2}\mathrm{i}I\right),A\right]_*+\left[\frac{1}{2}I\bullet\frac{1}{2}\mathrm{i}I,d_n(A)\right]_*+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}\left[d_m\left(\frac{1}{2}I\right)\bullet d_l\left(\frac{1}{2}\mathrm{i}I\right),d_j(A)\right]_*
\end{aligned}$$

$$=\left[\frac{1}{2}I\bullet\frac{1}{2}\mathrm{i}I,\ d_n(A)\right]_*=\mathrm{i}d_n(A).$$

Claim 10 For any $n\in\mathbb{N}$, d_n is additive.

Proof It follows from Claim 4 that d_1 is additive. Now assume d_k is additive for $k<n$. Then we complete the proof of Claim 10 in several steps.

Step 1 For any $n\in\mathbb{N}$, $A_{12}\in\mathcal{A}_{12}$ and $B_{21}\in\mathcal{A}_{21}$, we have $d_n(A_{12}+B_{21})=d_n(A_{12})+d_n(B_{21})$.

Set $T=d_n(A_{12}+B_{21})-d_n(A_{12})-d_n(B_{21})$. We only need to show $T=0$.

Since $\left[I\bullet(\mathrm{i}(P_2-P_1)),\ A_{12}\right]_*=\left[I\bullet(\mathrm{i}(P_2-P_1)),\ B_{21}\right]_*=0$, it follows that

$$\begin{aligned}0&=d_n(\left[I\bullet(\mathrm{i}(P_2-P_1)),\ A_{12}+B_{21}\right]_*)\\&=\left[d_n(I)\bullet(\mathrm{i}(P_2-P_1)),\ A_{12}+B_{21}\right]_*+\left[I\bullet d_n(\mathrm{i}(P_2-P_1)),\ A_{12}+B_{21}\right]_*+\left[I\bullet(\mathrm{i}(P_2-P_1)),\ d_n(A_{12}+B_{21})\right]_*\\&\quad+\sum_{\substack{m+l+j=n\\0\leqslant m,\ l,\ j\leqslant n-1}}\left[d_m(I)\bullet d_l\big(\mathrm{i}(P_2-P_1)\big),\ d_j\big(A_{12}+B_{21}\big)\right]_*.\end{aligned}$$

On the other hand,

$$\begin{aligned}0&=d_n(\left[I\bullet(\mathrm{i}(P_2-P_1)),\ A_{12}\right]_*)+d_n(\left[I\bullet(\mathrm{i}(P_2-P_1)),\ B_{21}\right]_*)\\&=\left[d_n(I)\bullet(\mathrm{i}(P_2-P_1)),\ A_{12}+B_{21}\right]_*+\left[I\bullet d_n(\mathrm{i}(P_2-P_1)),\ A_{12}+B_{21}\right]_*+\left[I\bullet(\mathrm{i}(P_2-P_1)),\ d_n(A_{12})+d_n(B_{21})\right]_*\\&\quad+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}\left[d_m(I)\bullet d_l\big(\mathrm{i}(P_2-P_1)\big),\ d_j\big(A_{12}+B_{21}\big)\right]_*.\end{aligned}$$

From two equations above, we get $\left[I\bullet(\mathrm{i}(P_2-P_1)),\ T\right]_*=0$. So $T_{11}=T_{22}=0$.

Since $\left[I\bullet A_{12},P_1\right]_*=0$, it follows that

$$\begin{aligned}d_n(\left[I\bullet(A_{12}+B_{21}),P_1\right]_*)&=\left[d_n(I)\bullet(A_{12}+B_{21}),\ P_1\right]_*+\left[I\bullet d_n(A_{12}+B_{21}),\ P_1\right]_*+\left[I\bullet(A_{12}+B_{21}),\ d_n(P_1)\right]_*\\&\quad+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}\left[d_m(I)\bullet d_l\big(A_{12}+B_{21}\big),d_j\big(P_1\big)\right]_*.\end{aligned}$$

On the other hand,

$$\begin{aligned}d_n(\left[I\bullet(A_{12}+B_{21}),\ P_1\right]_*)&=d_n(\left[I\bullet A_{12},\ P_1\right]_*)+d_n(\left[I\bullet B_{21},\ P_1\right]_*)\\&=\left[d_n(I)\bullet(A_{12}+B_{21}),\ P_1\right]_*+\left[I\bullet(d_n(A_{12})+d_n(B_{21})),\ P_1\right]_*+\left[I\bullet(A_{12}+B_{21}),\ d_n(P_1)\right]_*\\&\quad+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}\left[d_m(I)\bullet d_l\big(A_{12}\big)+d_l\big(B_{21}\big),\ d_j\big(P_1\big)\right]_*\\&=\left[d_n(I)\bullet(A_{12}+B_{21}),\ P_1\right]_*+\left[I\bullet(d_n(A_{12})+d_n(B_{21})),\ P_1\right]_*+\left[I\bullet(A_{12}+B_{21}),\ d_n(P_1)\right]_*\\&\quad+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}\left[d_m(I)\bullet d_l\big(A_{12}+B_{21}\big),\ d_j\big(P_1\big)\right]_*.\end{aligned}$$

Comparing two equations above, we can get $\left[I\bullet T,\ P_1\right]_*=0$. So $T_{21}=0$. Similarly, we can show $T_{12}=0$ by using $\left[I\bullet B_{21},\ P_2\right]_*=0$, then the step is proved.

Step 2 For any $n\in\mathbb{N}$, $A_{11}\in\mathcal{A}_{11}$, $B_{12}\in\mathcal{A}_{12}$, $C_{21}\in\mathcal{A}_{21}$ and $D_{22}\in\mathcal{A}_{22}$, we have $d_n(A_{11}+B_{12}+C_{21})=d_n(A_{11})+d_n(B_{12})+d_n(C_{21})$ and $d_n(B_{12}+C_{21}+D_{22})=d_n(B_{12})+d_n(C_{21})+d_n(D_{22})$.

Let $T=d_n(A_{11}+B_{12}+C_{21})-d_n(A_{11})-d_n(B_{12})-d_n(C_{21})$. It follows from Step 1 that

$$\begin{aligned}&d_n(\left[I\bullet(\mathrm{i}P_2),A_{11}+B_{12}+C_{21}\right]_*)\\&=\left[d_n(I)\bullet(\mathrm{i}P_2),A_{11}+B_{12}+C_{21}\right]_*+\left[I\bullet d_n(\mathrm{i}P_2),A_{11}+B_{12}+C_{21}\right]_*+\left[I\bullet(\mathrm{i}P_2),d_n(A_{11}+B_{12}+C_{21})\right]_*\\&\quad+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}\left[d_m(I)\bullet d_l\big(\mathrm{i}P_2\big),d_j\big(A_{11}+B_{12}+C_{21}\big)\right]_*\end{aligned}$$

$$= \left[d_n(I) \bullet (\mathrm{i}P_2), A_{11}+B_{12}+C_{21} \right]_* + \left[I \bullet d_n(\mathrm{i}P_2), A_{11}+B_{12}+C_{21} \right]_* + \left[I \bullet (\mathrm{i}P_2), d_n(A_{11}+B_{12}+C_{21}) \right]_*$$
$$+ \sum_{\substack{m+l+j=n \\ 0\leqslant m,l,j\leqslant n-1}} \left[d_m(I) \bullet d_l(\mathrm{i}P_2), d_j(A_{11}) + d_j(B_{12}) + d_j(C_{21}) \right]_*.$$

On the other hand,

$$d_n(\left[I \bullet (\mathrm{i}P_2),\ A_{11}+B_{12}+C_{21} \right]_*) = d_n(\left[I \bullet (\mathrm{i}P_2),\ A_{11} \right]_*) + d_n(\left[I \bullet (\mathrm{i}P_2),\ B_{12} \right]_*) + d_n(\left[I \bullet (\mathrm{i}P_2),\ C_{21} \right]_*)$$
$$= \left[d_n(I) \bullet (\mathrm{i}P_2), A_{11}+B_{12}+C_{21} \right]_* + \left[I \bullet d_n(\mathrm{i}P_2), A_{11}+B_{12}+C_{21} \right]_*$$
$$+ \left[I \bullet (\mathrm{i}P_2), d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21}) \right]_* + \sum_{\substack{m+l+j=n \\ 0\leqslant m,\ l,\ j\leqslant n-1}} \left[d_m(I) \bullet d_l(\mathrm{i}P_2), d_j(A_{11}) + d_j(B_{12}) + d_j(C_{21}) \right]_*.$$

From these equations, we get $\left[I \bullet (\mathrm{i}P_2), T \right]_* = 0$. So $T = T_{11}$.

Since $\left[I \bullet (\mathrm{i}(P_2 - P_1)), B_{12} \right]_* = \left[I \bullet (\mathrm{i}(P_2 - P_1)), C_{21} \right]_* = 0$, it follows that

$$d_n(\left[I \bullet (\mathrm{i}(P_2 - P_1)),\ A_{11}+B_{12}+C_{21} \right]_*)$$
$$= \left[d_n(I) \bullet (\mathrm{i}(P_2 - P_1)),\ A_{11}+B_{12}+C_{21} \right]_* + \left[I \bullet d_n(\mathrm{i}(P_2 - P_1)),\ A_{11}+B_{12}+C_{21} \right]_*$$
$$+ \left[I \bullet (\mathrm{i}(P_2 - P_1)),\ d_n(A_{11}+B_{12}+C_{21}) \right]_* + \sum_{\substack{m+l+j=n \\ 0\leqslant m,\ l,\ j\leqslant n-1}} \left[d_m(I) \bullet d_l\left(\mathrm{i}(P_2 - P_1)\right),\ d_j\left(A_{11}+B_{12}+C_{21}\right) \right]_*$$
$$= \left[d_n(I) \bullet (\mathrm{i}(P_2 - P_1)),\ A_{11}+B_{12}+C_{21} \right]_* + \left[I \bullet d_n(\mathrm{i}(P_2 - P_1)),\ A_{11}+B_{12}+C_{21} \right]_*$$
$$+ \left[I \bullet (\mathrm{i}(P_2 - P_1)),\ d_n(A_{11}+B_{12}+C_{21}) \right]_* + \sum_{\substack{m+l+j=n \\ 0\leqslant m,\ l,\ j\leqslant n-1}} \left[d_m(I) \bullet d_l\left(\mathrm{i}(P_2 - P_1)\right), d_j(A_{11}) + d_j(B_{12}) + d_j(C_{21}) \right]_*.$$

On the other hand,

$$d_n(\left[I \bullet (\mathrm{i}(P_2 - P_1)),\ A_{11}+B_{12}+C_{21} \right]_*)$$
$$= d_n(\left[I \bullet (\mathrm{i}(P_2 - P_1)),\ A_{11} \right]_*) + d_n(\left[I \bullet (\mathrm{i}(P_2 - P_1)),\ B_{12} \right]_*) + d_n(\left[I \bullet (\mathrm{i}(P_2 - P_1)),\ C_{21} \right]_*)$$
$$= \left[d_n(I) \bullet (\mathrm{i}(P_2 - P_1)),\ A_{11}+B_{12}+C_{21} \right]_* + \left[I \bullet d_n(\mathrm{i}(P_2 - P_1)),\ A_{11}+B_{12}+C_{21} \right]_*$$
$$+ \left[I \bullet (\mathrm{i}(P_2 - P_1)), d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21}) \right]_* + \sum_{\substack{m+l+j=n \\ 0\leqslant m,l,j\leqslant n-1}} \left[d_m(I) \bullet d_l\left(\mathrm{i}(P_2 - P_1)\right), d_j(A_{11}) + d_j(B_{12}) + d_j(C_{21}) \right]_*.$$

We can get $\left[I \bullet (\mathrm{i}(P_2 - P_1)),\ T \right]_* = 0$ from the equations above. So $T_{11} = 0$ and $T = 0$. Similarly, $d_n(B_{12}+C_{21}+D_{22}) = d_n(B_{12}) + d_n(C_{21}) + d_n(D_{22})$.

Step 3 For any $n \in \mathbb{N}$, $A_{11} \in \mathcal{A}_{11}$, $B_{12} \in \mathcal{A}_{12}$, $C_{21} \in \mathcal{A}_{21}$ and $D_{22} \in \mathcal{A}_{22}$, we have

$$d_n(A_{11}+B_{12}+C_{21}+D_{22}) = d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21}) + d_n(D_{22}).$$

Let $T = d_n(A_{11}+B_{12}+C_{21}+D_{22}) - d_n(A_{11}) - d_n(B_{12}) - d_n(C_{21}) - d_n(D_{22})$. It follows from Step 2 that

$$d_n(\left[I \bullet (\mathrm{i}P_2),\ A_{11}+B_{12}+C_{21}+D_{22} \right]_*)$$
$$= \left[d_n(I) \bullet (\mathrm{i}P_2),\ A_{11}+B_{12}+C_{21}+D_{22} \right]_* + \left[I \bullet d_n(\mathrm{i}P_2),\ A_{11}+B_{12}+C_{21}+D_{22} \right]_*$$
$$+ \left[I \bullet (\mathrm{i}P_2),\ d_n(A_{11}+B_{12}+C_{21}+D_{22}) \right]_* + \sum_{\substack{m+l+j=n \\ 0\leqslant m,l,j\leqslant n-1}} \left[d_m(I) \bullet d_l(\mathrm{i}P_2),\ d_j\left(A_{11}+B_{12}+C_{21}+D_{22}\right) \right]_*$$
$$= \left[d_n(I) \bullet (\mathrm{i}P_2),\ A_{11}+B_{12}+C_{21}+D_{22} \right]_* + \left[I \bullet d_n(\mathrm{i}P_2),\ A_{11}+B_{12}+C_{21}+D_{22} \right]_*$$
$$+ \left[I \bullet (\mathrm{i}P_2),\ d_n(A_{11}+B_{12}+C_{21}+D_{22}) \right]_* + \sum_{\substack{m+l+j=n \\ 0\leqslant m,l,j\leqslant n-1}} \left[d_m(I) \bullet d_l(\mathrm{i}P_2),\ d_j(A_{11}) + d_j(B_{12}) + d_j(C_{21}) + d_j(D_{22}) \right]_*.$$

On the other hand,

$$d_n(\left[I \bullet (\mathrm{i}P_2), A_{11}+B_{12}+C_{21}+D_{22} \right]_*)$$
$$= d_n(\left[I \bullet (\mathrm{i}P_2), A_{11} \right]_*) + d_n(\left[I \bullet (\mathrm{i}P_2), B_{12} \right]_*) + d_n(\left[I \bullet (\mathrm{i}P_2), C_{21} \right]_*) + d_n(\left[I \bullet (\mathrm{i}P_2), D_{22} \right]_*)$$

$$=\left[d_n(I)\bullet(\mathrm{i}P_2),A_{11}+B_{12}+C_{21}+D_{22}\right]_*+\left[I\bullet d_n(\mathrm{i}P_2),A_{11}+B_{12}+C_{21}+D_{22}\right]_*$$
$$+\left[I\bullet(\mathrm{i}P_2),d_n(A_{11})+d_n(B_{12})+d_n(C_{21})+d_n(D_{22})\right]_*+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}\left[d_m(I)\bullet d_l(\mathrm{i}P_2),d_j(A_{11})+d_j(B_{12})+d_j(C_{21})+d_j(D_{22})\right]_*.$$

Comparing the two equations above, we can get $\left[I\bullet(\mathrm{i}P_2),\ T\right]_*=0$. So $T_{12}=T_{21}=T_{22}=0$. Similarly, we can show that $T_{11}=0$ from $\left[I\bullet(\mathrm{i}P_1),D_{22}\right]_*=0$, then the Step 3 is proved.

Step 4 For any $n\in\mathbb{N}$, $A_{st},\ B_{st}\in\mathcal{A}_{st},\ 1\leqslant s\neq t\leqslant 2$, we have $d_n(A_{st}+B_{st})=d_n(A_{st})+d_n(B_{st})$.

Since $\left[\frac{1}{2}I\bullet(P_s+A_{st}),P_t+B_{st}\right]_*=A_{st}+B_{st}-A_{st}{}^*-B_{st}A_{st}{}^*$, we can get

$$d_n(A_{st}+B_{st})+d_n(-A_{st}{}^*)+d_n(-B_{st}A_{st}{}^*)=d_n\left(\left[\frac{1}{2}I\bullet(P_s+A_{st}),\ P_t+B_{st}\right]_*\right)$$
$$=\left[d_n(\frac{1}{2}I)\bullet(P_s+A_{st}),\ P_t+B_{st}\right]_*+\left[\frac{1}{2}I\bullet d_n(P_s+A_{st}),\ P_t+B_{st}\right]_*+\left[\frac{1}{2}I\bullet(P_s+A_{st}),\ d_n(P_t+B_{st})\right]_*$$
$$+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}\left[d_m(\frac{1}{2}I)\bullet d_l(P_s+A_{st}),\ d_j(P_t+A_{st})\right]_*$$
$$=d_n\left(\left[\frac{1}{2}I\bullet P_s,P_t\right]_*\right)+d_n\left(\left[\frac{1}{2}I\bullet A_{st},P_t\right]_*\right)+d_n\left(\left[\frac{1}{2}I\bullet P_s,B_{st}\right]_*\right)+d_n\left(\left[\frac{1}{2}I\bullet A_{st},B_{st}\right]_*\right)$$
$$=d_n(A_{st})+d_n(-A_{st}{}^*)+d_n(B_{st})+d_n(-B_{st}A_{st}{}^*).$$

Step 5 For any $n\in\mathbb{N}$, $A_{ss},B_{ss}\in\mathcal{A}_{ss},\ s=1,2$, we have $d_n(A_{ss}+B_{ss})=d_n(A_{ss})+d_n(B_{ss})$.

Let $T=d_n(A_{ss}+B_{ss})-d_n(A_{ss})-d_n(B_{ss})$. For $1\leqslant s\neq t\leqslant 2$, it follows that

$$\left[d_n(I)\bullet P_t,\ A_{ss}+B_{ss}\right]_*+\left[I\bullet d_n(P_t),\ A_{ss}+B_{ss}\right]_*+\left[I\bullet P_t,\ d_n(A_{ss}+B_{ss})\right]_*+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}\left[d_m(I)\bullet d_l(P_t),\ d_j(A_{ss}+B_{ss})\right]_*$$
$$=d_n(\left[I\bullet P_t,\ A_{ss}+B_{ss}\right]_*)=d_n(\left[I\bullet P_t,\ A_{ss}\right]_*)+d_n(\left[I\bullet P_t,\ B_{ss}\right]_*)$$
$$=\left[d_n(I)\bullet P_t,A_{ss}+B_{ss}\right]_*+\left[I\bullet d_n(P_t),A_{ss}+B_{ss}\right]_*+\left[I\bullet P_t,d_n(A_{ss})+d_n(B_{ss})\right]_*+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}\left[d_m(I)\bullet d_l(P_t),d_j(A_{ss}+B_{ss})\right]_*.$$

From the equation above, we can get $T=T_{ss}$.

For all $C_{st}\in\mathcal{A}_{st},\ s\neq t$, it follows from Step 4 that

$$\left[d_n(I)\bullet(A_{ss}+B_{ss}),C_{st}\right]_*+\left[I\bullet d_n(A_{ss}+B_{ss}),C_{st}\right]_*+\left[I\bullet(A_{ss}+B_{ss}),d_n(C_{st})\right]_*+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}\left[d_m(I)\bullet d_l(A_{ss}+B_{ss}),d_j(C_{st})\right]_*$$
$$=d_n(\left[I\bullet(A_{ss}+B_{ss}),C_{st}\right]_*)=d_n(\left[I\bullet A_{ss},C_{st}\right]_*)+d_n(\left[I\bullet B_{ss},C_{st}\right]_*)$$
$$=\left[d_n(I)\bullet(A_{ss}+B_{ss}),C_{st}\right]_*+\left[I\bullet d_n(A_{ss})+d_n(B_{ss}),C_{st}\right]_*+\left[I\bullet(A_{ss}+B_{ss}),d_n(C_{st})\right]_*+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}\left[d_m(I)\bullet d_l(A_{ss}+B_{ss}),d_j(C_{st})\right]_*.$$

Hence $\left[I\bullet T_{ss},C_{st}\right]=0$ for all $C_{st}\in\mathcal{A}_{st}$, then $T_{ss}CP_t=0$ for all $C\in\mathcal{A}$. So $T=T_{ss}=0$, then the step is proved.

Now, it follows from Steps 3-5 that d_n is additive, and Claim 10 is proved.

Claim 11 For any $A\in\mathcal{A}$ and $n\in\mathbb{N}$, $d_n(A^*)=d_n(A)^*$.

Proof For any $n\in\mathbb{N}$, $A\in\mathcal{A}$, where $A=A_1+\mathrm{i}A_2$, $A_1,\ A_2\in\mathcal{S}$, it follows from Claims 8-10 that

$$d_n(A^*)=d_n(A_1-\mathrm{i}A_2)=d_n(A_1)-\mathrm{i}d_n(A_2)=d_n(A_1)^*+\left(\mathrm{i}d_n(A_2)\right)^*=d_n(A_1+\mathrm{i}A_2)^*=d_n(A)^*.$$

Claim 12 For any $n\in\mathbb{N}$, d_n is an additive *-higher derivation on $\mathcal{A}$.

Proof To complete the proof, we only need to show that d_n is a higher derivation on $\mathcal{A}$. Since d_n is additive. $[A,B]_*=AB-BA^*$ and $[\mathrm{i}A,\ \mathrm{i}B]_*=-AB-BA^*$ for all $A,B\in\mathcal{A}$. It follows from Claim 9 that

$$d_n(AB)-d_n(BA^*)=d_n([A,B]_*)=d_n\left(\left[\frac{1}{2}I\bullet A,B\right]_*\right)$$

$$=\left[d_n\left(\frac{1}{2}I\right)\bullet A,B\right]_*+\left[\frac{1}{2}I\bullet d_n(A),B\right]_*+\left[\frac{1}{2}I\bullet A,d_n(B)\right]_*+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}\left[d_m\left(\frac{1}{2}I\right)\bullet d_l(A),d_j(B)\right]_*$$

$$=\left[\frac{1}{2}I\bullet d_n(A),B\right]_*+\left[\frac{1}{2}I\bullet A,d_n(B)\right]_*+\sum_{\substack{l+j=n\\0\leqslant l,j\leqslant n-1}}\left[d_l(A),d_j(B)\right]_*$$

$$=d_n(A)B-Bd_n(A)^*+Ad_n(B)-d_n(B)A^*+\sum_{\substack{l+j=n\\0\leqslant l,j\leqslant n-1}}d_l(A)d_j(B)-d_j(B)d_l(A)^*.$$

And

$$-d_n(AB)-d_n(BA^*)=d_n([\mathrm{i}A,\mathrm{i}B]_*)=d_n\left(\left[\frac{1}{2}I\bullet \mathrm{i}A,\mathrm{i}B\right]_*\right)$$

$$=\left[d_n\left(\frac{1}{2}I\right)\bullet \mathrm{i}A,\mathrm{i}B\right]_*+\left[\frac{1}{2}I\bullet d_n(\mathrm{i}A),\mathrm{i}B\right]_*+\left[\frac{1}{2}I\bullet \mathrm{i}A,d_n(\mathrm{i}B)\right]_*+\sum_{\substack{m+l+j=n\\0\leqslant m,l,j\leqslant n-1}}\left[d_m\left(\frac{1}{2}I\right)\bullet d_l(\mathrm{i}A),d_j(\mathrm{i}B)\right]_*$$

$$=\left[\frac{1}{2}I\bullet d_n(\mathrm{i}A),\mathrm{i}B\right]_*+\left[\frac{1}{2}I\bullet \mathrm{i}A,d_n(\mathrm{i}B)\right]_*+\sum_{\substack{l+j=n\\0\leqslant l,j\leqslant n-1}}\left[d_l(\mathrm{i}A),d_j(\mathrm{i}B)\right]_*$$

$$=-d_n(A)B-Bd_n(A)^*-Ad_n(B)-d_n(B)A^*+\sum_{\substack{l+j=n\\0\leqslant l,j\leqslant n-1}}-d_l(A)d_j(B)-d_j(B)d_l(A)^*.$$

Using the two equations above, we obtain $d_n(AB)=d_n(A)B+Ad_n(B)+\sum_{\substack{l+j=n\\0\leqslant l,j\leqslant n-1}}d_l(A)d_j(B)=\sum_{l+j=n}d_l(A)d_j(B)$, then Claim 12 is proved.

It follows from Claim 10-12 that $\mathcal{D}=\{d_n\}_{n\in\mathbb{N}}$ is an additive *-higher derivation on $\mathcal{A}$, which completes the proof of Theorem 2. Obviously, prime $*$-algebras satisfy (1) and (2), we have the following corollary.

Corollary 1 Let $\mathcal{A}$ be a prime *-algebra with unit I and P be a nontrivial projection in $\mathcal{A}$. Then any nonlinear mixed Jordan triple *-higher derivation $\mathcal{D}=\{d_n\}_{n\in\mathbb{N}}$, d_n: $\mathcal{A}\to\mathcal{A}$ is an additive *-higher derivation.

A von Neumann algebra $\mathcal{M}$ is a weekly closed, self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator I. It is shown in Ref. [3] and Ref. [14] that if a von Neumann algebra has no central summands of type I_1, then $\mathcal{M}$ satisfies (1) and (2). Now we have following corollary.

Corollary 2 Let $\mathcal{M}$ be a von Neumann algebra with no central summands of type I_1. Then any nonlinear mixed Jordan triple *-higher derivation $\mathcal{D}=\{d_n\}_{n\in\mathbb{N}}$, d_n: $\mathcal{A}\to\mathcal{A}$ is an additive *-higher derivation.

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*-代数上的非线性混合 Jordan 三重 *-高阶导子

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摘要：设 $\mathcal{A}$ 是一个包含非平凡投影和单位元的 *-代数，ℕ 是非负整数集。在满足一定条件的 $\mathcal{A}$ 上，我们证明了任意非线性混合 Jordan 三重 *-高阶导子 $\mathcal{D}=\{d_n\}_{n\in\mathbb{N}}$ 都是可加的 *-高阶导子。同时作为应用，我们将这一结论运用到了素 *-代数和无 I_1 型中心直和项的 von Neumann 代数。

关键词：混合 Jordan 三重 *-高阶导子；*-高阶导子；von Neumann 代数

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