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The Existence of Solution to the Even Orlicz Chord Minkowski Problem

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Abstract: Chord measures are newly discovered translation-invariant geometric measures of convex bodies in $\mathbb{R}^n$ by Lutwak-Xi-Yang-Zhang (*Communications on Pure and Applied Mathematics*, 2024), which is an extension of the surface area measure. The Minkowski problems for chord measures was considered by Lutwak-Xi-Yang-Zhang. In this paper, we use variational method to solve the even Orlicz chord Minkowski problem. The obtained results are an extension of the even Orlicz Minkowski problem from Haberl-Lutwak-Yang-Zhang (*Advances in Mathematics*, 2010).

Key words: chord integral; chord measure; Minkowski problem

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0 Introduction

The classical Brunn-Minkowski theory is the core of convex geometry analysis. The classical Minkowski problem asks for the existence, uniqueness, and regularity of a convex body whose surface area measure is equal to a pre-given Borel measure. Lutwak[1-2] developed the classical Brunn-Minkowski theory into L_p-Brunn-Minkowski theory. After that, the L_p Minkowski problem and related researches can be found in Refs.[3-14].

The L_p Minkowski problem has been extended to the even Orlicz Minkowski problem by Habrel *et al*[15] with a series of papers on the Orlicz Minkowski theory [16-17]. The Orlicz Minkowski problem and related topics can be found in the Refs.[18-29].

Recently, Lutwak *et al*[30] introduced a new family of geometric measures by studying a variational formula on the geometric invariants of convex body integrals, which called chord integrals. Accordingly, the L_p chord Minkowski problem was considered. Xi *et al*[31] solved the L_p chord Minkowski problem when $p>1$, $q>1$ and the symmetric case of $0<p<1$. Guo *et al*[32] solved the L_p chord Minkowski problem for $0\leqslant p<1$ without symmetric assumptions. Li[33] solved the discrete L_p chord Minkowski problem in the condition of $p<0$ and $q>0$, and as for general Borel measure, Li also gave a proof but need $-n<p<0$ and $1<q<n+1$. Hu *et al* [34] used flow methods to get regularity of the chord log-Minkowski problem of $p=0$. In addition, Hu *et al*[35] also found the smooth origin-symmetric solution for L_p chord Minkowski problem in the case of $\{p>0,q>3\}\cup$

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$\{-n<p<0, 3<q<n+1\}$ by using same flow method[34]. Zhao *et al*[36] generalized the L_p chord Minkowski problem and sloved the existence of smooth solutions to the Orlicz chord Minkowski problem.

In this paper, we use variational method to solve the even Orlicz chord Minkowski problem.

Let $\mathcal{K}^n$ be the collection of convex bodies (compact convex sets with nonempty interior) in $\mathbb{R}^n$. For $K\in\mathcal{K}^n$, the chord integral $I_q(K)$ of K is defined as follows:

$$I_q(K)=\int_{\mathcal{L}^n}|K\cap\ell|^q\,\mathrm{d}\ell,\quad q\geqslant 0,$$

where $|K\cap\ell|$ denotes the length of the chord $|K\cap\ell|$, and the integration is with respect to the Haar measure on the Grassmannian $\mathcal{L}^n$ of lines in $\mathbb{R}^n$.

Chord integrals contain volume $V(K)$ and surface area $S(K)$ as two important special cases:

$$I_1(K)=V(K),\ I_0(K)=\frac{\omega_{n-1}}{n\omega_n}S(K),\ I_{n+1}(K)=\frac{n+1}{\omega_n}V(K)^2,$$

where ω_n is the volume enclosed by the unit sphere S^{n-1}.

We can see in Ref.[15] that the differential of $I_q(K)$ defines a finite Borel measure $F_q(K,\cdot)$ on S^{n-1}. Precisely, for convex bodies K and L in $\mathbb{R}^n$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0^+}I_q(K+tL)=\int_{S^{n-1}}h_L(v)\mathrm{d}F_q(K,v),\quad q\geqslant 0,$$

where $F_q(K,\cdot)$ is called the q-th chord measure of K and h_L is the support function of L. The cases of $q=0,1$ of this formula are classical, which are the variational formulas of surface area and volume.

$$F_0(K,\cdot)=\frac{(n-1)\omega_{n-1}}{n\omega_n}S_{n-2}(K,\cdot),\ F_1(K,\cdot)=S_{n-1}(K,\cdot),$$

where $S_{n-2}(K,\cdot)$ and $S_{n-1}(K,\cdot)$ are respecting the $(n-2)$-th order and $(n-1)$-th order area measure of K.

The Orlicz chord Minkowski problem was stated in Ref.[36] by the following form:

The Orlicz chord Minkowski problem: Suppose $\varphi:(0,\infty)\to(0,\infty)$ is a continuous decreasing function. If μ is an finite Borel measure on S^{n-1} which is not concentrated on a great subsphere of S^{n-1}, what are the necessary and sufficient conditions on μ , such that there exist a convex body $K\in\mathcal{K}_o^n$ in $\mathbb{R}^n$ and a positive constant c so that

$$\mu=c\varphi(h_K)\mathrm{d}F_q(K,\cdot)\,?$$

Remark 1 When $\varphi(s)=s^{1-p}$, the Orlicz chord Minkowski problem is reduce to the L_p chord Minkowski problem[30-31]. When $q=1$, the Orlicz chord Minkowski problem is reduced to the Orlicz Minkowski problem[15].

In this paper, we consider the even Orlicz chord Minkowski problem. Concretely, we prove the following theorem.

Theorem 1 *L*et $q>1$ and $0<\alpha<1$. Suppose $\varphi:(0,\infty)\to(0,\infty)$ is a continuous decreasing function. If μ is an even Borel measure on S^{n-1} that is not concentrated on a great subsphere of S^{n-1}, then there exists a symmetric, convex body K_0 such that

$$c\varphi(h_{K_0})\mathrm{d}F_q(K_0,\cdot)=\mathrm{d}\mu,$$

with $c=I_q(K_0)^{\frac{\alpha-(n+q-1)}{n+q-1}}$.

This paper is organized as follows. In Section 1, we introduce some basic facts about convex bodies. In Section 2, we give some lemmas needed for the proof of theorems in Section 3. In Section 3, we prove the main theorem.

1 Preliminaries

In this section, we will give some relative notations and facts about convex bodies. And for more details, see Refs.[37-39].

Let $\mathbb{R}^n$ be n-dimensional Euclidean space. The standard inner product in $\mathbb{R}^n$ is denoted by $x\cdot y$. We write $S^{n-1}=\{x\in\mathbb{R}^n: x\cdot x=1\}$ for the boundary of the Euclidean unit ball B in $\mathbb{R}^n$.

A *convex body* is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. The set of convex bodies in $\mathbb{R}^n$ containing the origin in their interiors is denoted by $\mathcal{K}_o^n$. The set of convex bodies in $\mathbb{R}^n$ that are symmetric about the origin will be denoted by $\mathcal{K}_e^n$.

A compact, convex set $K\in\mathbb{R}^n$ is uniquely determined by its *support function* $h_K:\mathbb{R}^n\to\mathbb{R}$, where $h_K(x)=\max\{x\cdot y: y\in K\}$, for each $x\in\mathbb{R}^n$. For example, the support function of the line segment $\bar{v}$ joining the points $\pm v\in\mathbb{R}^n$ is given by

$$h_{\bar{v}}(x)=|x\cdot v|,\ x\in\mathbb{R}^n.$$

It is trivial that for the support function of the dilate $cK=\{cx: x\in K\}$ of a compact, convex K , we have

$$h_{cK}=ch_K,\ c>0. \tag{1}$$

Note that support functions are positively homogeneous of degree 1 and subadditive. It follows immediately from the definition of support functions that for compact, convex $K, L\subset\mathbb{R}^n$,

$$K\subseteq L\Leftrightarrow h_K\leqslant h_L. \tag{2}$$

Consequently, the support function of a body $K\in\mathcal{K}_o^n$ is bounded from above and below by positive reals.

The set of continuous functions on the sphere S^{n-1} will be denoted by $C(S^{n-1})$ and will always be viewed as equipped with the max-norm metric:

$$\| f-g \|_\infty = \max_{u\in S^{n-1}} | f(u)-g(u) |,$$

for $f,\ g \in C(S^{n-1})$. The subspace of positive continuous functions will be denoted by $C^+(S^{n-1})$ and the subspace of $C^+(S^{n-1})$ consisting of only the even functions will be denoted by $C_e^+(S^{n-1})$.

For $h \in C(S^{n-1})$, the Wulff-shape $[h]$ is a compact convex set defined by

$$[h] = \{ x \in \mathbb{R}^n : x \cdot v \leqslant h(v), \forall v \in S^{n-1} \}.$$

Clearly, $h_{[h]}(v) \leqslant h(v)$.

The set $\mathcal{K}_o^n$ will be viewed as equipped with the Hausdorff metric. If a sequence K_i of bodies in $\mathcal{K}_o^n$ and a body $K \in \mathcal{K}_o^n$, we say that $\lim\limits_{i\to\infty} K_i = K$ provided

$$\| h_{K_i} - h_K \|_\infty \to 0.$$

If $x \cdot u = h_K(u)$, a boundary point $x \in \partial K$ has $u \in S^{n-1}$ as an outer normal. A boundary point is said to be singular if it has more than one unit normal vector. It is well known that the set of singular boundary points of a convex body has $\mathcal{H}^{n-1}$-measure equal to 0.

The inverse spherical image of a convex body K at $\omega \subset S^{n-1}$ which is a Borel set, denoted by $\tau(K,\omega)$, is the set of all boundary points of K which have an outer unit normal belonging to the set ω. S_K which is called the Aleksandrov-Fenchel-Jensen *surface area measure* of K is defined by $S_K(\omega) = \mathcal{H}^{n-1}(\tau(K,\omega))$, for each Borel set $\omega \subset S^{n-1}$, associated with each convex body $K \in \mathcal{K}_o^n$ is a Borel measure on S^{n-1}.

2 Some Lemmas

This section mainly gives some lemmas needed for the proof in Section 3. First, we give the properties of a simple functional.

Lemma 1[15] Suppose $\varphi: (0,\infty) \to (0,\infty)$ is a continuous decreasing function. Define the function $\phi: [0,\infty) \to [0,\infty)$ by $\phi(t) = \int_0^t \frac{1}{\varphi(s)} \mathrm{d}s$. For $c>0$ and $0<\alpha<1$, we have

$$\lim_{t\to\infty} \frac{\phi(ct)}{t^\alpha} = \infty \tag{3}$$

and

$$\lim_{t\to 0^+} \frac{\phi(t)}{t^\alpha} = 0. \tag{4}$$

Moreover, the derivative of ϕ is increasing and the function ϕ is convex.

The following lemma is a slight variant of a standard result about differentiability under an integral sign.

Lemma 2[15] Let $\phi: (0,\infty) \to (0,\infty)$ be continuously differentiable, $I \subset \mathbb{R}$ be an open interval, and

$$h: I \times S^{n-1} \to (0,\infty),\ (t,u) \mapsto h(t,u)$$

be a continuous function such that the partial derivative $\frac{\partial h}{\partial t}(t,u)$ exists for all $(t,u) \in I \times S^{n-1}$. If $\frac{\partial h}{\partial t}$ is bounded and h is bounded from above and from below by positive numbers, then the function $H: I \to (0,\infty)$ defined by

$$H(t) = \int_{S^{n-1}} (\phi \circ h)(t,u) \mathrm{d}\mu(u),$$

for $t \in I$, is differentiable on I and

$$H'(t) = \int_{S^{n-1}} \frac{\partial(\phi \circ h)}{\partial t}(t,u) \mathrm{d}\mu(u).$$

Moreover, if $\partial(\phi \circ h)/\partial t$ is continuous with respect to t, then H' is continuous.

To solve the maximization problem, we use the variational formula in the following lemma.

Lemma 3[30] Let $q>0$. Suppose that $g: S^{n-1} \to \mathbb{R}$ is continuous and $h_t: S^{n-1} \to (0,\infty)$ is a family of continuous functions given as follows:

$$h_t = h_0 + tg + o(t,\cdot),$$

for each $t \in (-\delta,\delta)$ with $\delta>0$. Here, $o(t,\cdot) \in C(S^{n-1})$ and $o(t,\cdot)/t$ tends to 0 uniformly on S^{n-1} as $t \to 0$. Let K_t be the Wulff-shape generated by h_t and K be the Wulff-shape generated by h_0. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} I_q(K_t) = \int_{S^{n-1}} g(v) \mathrm{d}F_q(K,v).$$

3 The Even Orlicz Chord Minkowski Problem

Let $q \neq 0$ and $0<\alpha<1$. Suppose $\varphi: (0,\infty) \to (0,\infty)$ is a continuous decreasing function. Define the function ϕ: $[0,\infty) \to [0,\infty)$ by $\phi(t) = \int_0^t \frac{1}{\varphi(s)} \mathrm{d}s$. For each non-zero even finite Borel measure μ on S^{n-1}, define the functional $\Phi_{\phi,q}(h)$: $C^+(S^{n-1}) \to \mathbb{R}$ by

$$\Phi_{\phi,q}(h) = \frac{n+q-1}{\alpha} I_q([h])^{\frac{\alpha}{n+q-1}} - \int_{S^{n-1}} (\phi \circ h) \mathrm{d}\mu.$$

The following theorem shows that if the maximization can be obtained, the even Orlicz chord Minkowski problem has a solution.

Theorem 2 Let $q>0$ and μ be a nonzero even finite Borel measure on S^{n-1} that is not concentrated on a great subsphere of S^{n-1}. If the maximization problem

$$\sup\left\{\Phi_{\phi,q}(h)\colon h\in C^{+}(S^{n-1})\right\}$$

has a solution $h_0\in C^{+}(S^{n-1})$, then there exists $K_0\in\mathcal{K}_e^n$ such that $c\varphi(h_{K_0})\mathrm{d}F_q(K_0,\cdot)=\mathrm{d}\mu$.

Proof Define $h_t=h_0+tg$ for $g\in C(S^{n-1})$, $t\in(-\delta,\delta)$ for $\delta>0$. For sufficiently small $|t|$, the family $h_t\in C^{+}(S^{n-1})$. By Lemma 2 and Lemma 3, the function $t\to\Phi_{\phi,q}(h_t)$ is differentiable at 0. By the fact that h_0 is a maximizer,

$$\begin{aligned}0&=\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\phi,q}(h_t)|_{t=0}\\&=I_q([h_0])^{\frac{\alpha-(n+q-1)}{n+q-1}}\int_{S^{n-1}}g(v)\,\mathrm{d}F_q([h_0],v)\\&\quad-\int_{S^{n-1}}\frac{1}{\varphi(h_0)}g(v)\mathrm{d}\mu(v).\end{aligned}$$

Since $g\in C(S^{n-1})$ is arbitrary and using the fact that $h_0=h_{[h_0]}$ almost everywhere, for $F_q([h_0],\cdot)$, we have

$$I_q([h_0])^{\frac{\alpha-(n+q-1)}{n+q-1}}\varphi(h_{[h_0]})\mathrm{d}F_q([h_0],v)=\mathrm{d}\mu(v).$$

Let $[h_0]=K_0\in\mathcal{K}_e^n$, we have

$$c\varphi(h_{K_0})\mathrm{d}F_q(K_0,v)=\mathrm{d}\mu(v),$$

where $c=I_q(K_0)^{\frac{\alpha-(n+q-1)}{n+q-1}}$.

Next, we will prove the main theorem by showing that $\Phi_{\phi,q}$ attains a maximum .

Theorem 3 Let $q>0$ and $0<\alpha<1$. Suppose φ: $(0,\infty)\to(0,\infty)$ is a continuous decreasing function. If μ is an even finite Borel measure on S^{n-1} that is not concentrated on a great subsphere of S^{n-1}, then there exists a convex body $K_0\in\mathcal{K}_e^n$ such that

$$c\varphi\left(h_{K_0}\right)\mathrm{d}F_q\left(K_0,\cdot\right)=\mathrm{d}\mu$$

with $c=I_q(K_0)^{\frac{\alpha-(n+q-1)}{n+q-1}}$.

Proof For $s>0$, let $\psi_s\colon(0,\infty)\to\mathbb{R}$ be defined by

$$\psi_s(t)=\left(\frac{n+q-1}{\alpha}I_q(B)^{\frac{\alpha}{n+q-1}}-\frac{|\mu|\phi(st)}{t^{\alpha}}\right)t^{\alpha}.$$

From (3) we conclude that $\lim\limits_{t\to\infty}\psi_s(t)=-\infty$. In particular, for each s, there is a positive number $r_s>0$ such that

$$t>r_s\Rightarrow\psi_s(t)<0. \tag{5}$$

We use B to denote the centered unit ball in $\mathbb{R}^n$. For every $r>0$,

$$\begin{aligned}\Phi_{\phi,q}(h_{rB})&=\frac{n+q-1}{\alpha}r^{\alpha}I_q(B)^{\frac{\alpha}{n+q-1}}-\phi(r)|\mu|\\&=\left(\frac{n+q-1}{\alpha}I_q(B)^{\frac{\alpha}{n+q-1}}-\frac{|\mu|\phi(r)}{r^{\alpha}}\right)r^{\alpha}.\end{aligned}$$

By the equation of (4), it follows that $\Phi_{\phi,q}(h_{rB})$ is positive for small positive r.

Hence, there exists $K\in\mathcal{K}_e^n$ such that

$$\Phi_{\phi,q}(K)>0. \tag{6}$$

The Wulff-shape $[h]$ associated with a given function $h\in C_e^{+}(S^{n-1})$ is origin symmetric and has a support function $h_{[h]}$ with satisfies $0<h_{[h]}\leqslant h$. Since ϕ is increasing and $I_q(h_{[h]})=I_q(h)$, we deduce $\Phi_{\phi,q}(h)\leqslant\Phi_{\phi,q}(h_{[h]})$.

Next, we will show that the search for a function at which $\Phi_{\phi,q}$ attains a maximum can be further restricted to support functions of origin symmetric convex bodies contained in some ball of fixed radius. To this end, first note that the continuous function on S^{n-1},

$$v\mapsto\int_{S^{n-1}}h_{\bar{v}}\,\mathrm{d}\mu,$$

is positive since μ is not concentrated on a great subsphere. Thus, there exists a $s\in(0,\infty)$ such that

$$\frac{1}{|\mu|}\int_{S^{n-1}}h_{\bar{v}}\,\mathrm{d}\mu\geqslant s,\ \text{for every } v\in S^{n-1}. \tag{7}$$

Let $K\in\mathcal{K}_e^n$ and choose $v_K\in S^{n-1}$ such that for a suitable real $r_K>0$ the point r_Kv_K is an element of K with maximal distance from the origin. Since K is origin symmetric, the line segment with endpoints $\pm r_Kv_K$ is contained in K. From (3) and (2), we deduce $r_Kh_{\bar{v}K}\leqslant h_K$. The monotonicity of ϕ, Jensen's inequality, and (7) therefore yield

$$\begin{aligned}\int_{S^{n-1}}\phi\left(h_K\right)\mathrm{d}\mu&\geqslant\int_{S^{n-1}}\phi\left(r_Kh_{\bar{v}K}\right)\mathrm{d}\mu\\&\geqslant|\mu|\phi\left(\frac{1}{|\mu|}\int_{S^{n-1}}r_Kh_{\bar{v}K}\,\mathrm{d}\mu\right)\geqslant|\mu|\phi(s\cdot r_K).\end{aligned}$$

Since $K\subset r_KB$, the last inequality show that

$$\begin{aligned}\Phi_{\phi,q}(h_K)&=\frac{n+q-1}{\alpha}I_q([h_K])^{\frac{\alpha}{n+q-1}}-\int_{S^{n-1}}\phi\left(h_K\right)\mathrm{d}\mu\\&\leqslant\frac{n+q-1}{\alpha}r_k^{\alpha}I_q(B)^{\frac{\alpha}{n+q-1}}-|\mu|\phi(s\cdot r_K)\\&=\left(\frac{n+q-1}{\alpha}I_q(B)^{\frac{\alpha}{n+q-1}}-\frac{|\mu|\phi(s\cdot r_K)}{r_k^{\alpha}}\right)r_k^{\alpha}\\&=\psi_s(r_K).\end{aligned}$$

From (6) we therefore conclude that there exists a real $r=r_s>0$ such that

$$r_K>r\Rightarrow\Phi_{\phi,q}(K)<0. \tag{8}$$

It follows from (6) and (8) that in order to find a maximum of the functional $\Phi_{\phi,q}$ on $C_e^{+}(S^{n-1})$, it is sufficient to search among support functions of members of the set

$$\mathcal{F}=\left\{K\in\mathcal{K}_e^n\colon K\subset rB\right\}.$$

Let $\{K_i\}$ be a maximizing sequence in $\mathcal{F}$ for $\Phi_{\phi,q}$, i.e

$$\lim_{i\to\infty}\Phi_{\phi,q}(h_{K_i})=\sup\left\{\Phi_{\phi,q}(h_K):K\in\mathcal{F}\right\}.$$

Obviously, the sequence $\{K_i\}$ is bounded. By the Blaschke's selection theorem, there exists a convergent subsequence, which we also denote by $\{K_i\}$, with $\lim_{i\to\infty}K_i=K_0$, where K_i is origin symmetric. Hence, K_0 is origin symmetric. Let $h_0=h_{K_o}$ and by the continuity of functional $\Phi_{\phi,q}$, we deduce that

$$\frac{n+q-1}{\alpha}I_q([h_0])^{\frac{\alpha}{n+q-1}}\geqslant\Phi_{\phi,q}(h_0)=\Phi_{\phi,q}(h_{K_0})=\lim_{i\to\infty}\Phi_{\phi,q}(h_{K_i})$$
$$=\sup\left\{\Phi_{\phi,q}(h_K):K\in\mathcal{F}\right\}>0.$$

Consequently, K_0 has non-empty interior and thus $K_0\in\mathcal{K}_e^n$. Therefore, h_0 is a solution of the maximization problem for Theorem 2. This completes the proof.

Remark 2 When $\varphi(s)=s^{1-p}$, Theorem 3 is reduce to the Theorem 4.2 of Ref. [31] in the case of $p>1$. When $q=1$, Theorem 1 is reduced to the Theorem 1 of Ref.[15].

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偶 Orlicz 弦 Minkowski 问题解的存在性

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摘要：2024年，Lutwak等人引入凸体的弦测度 (*Communications on Pure and Applied Mathematics*, 2024)。经典的表面积测度是一种特殊的弦测度。关于弦测度的Minkowski问题是凸几何中新的研究热点。本文采用变分方法研究 Orlicz 情形下偶弦测度的 Minkowski 问题解的存在性，所得到的结果是 Haberl-Lutwak-Yang-Zhang (*Advances in Mathematics*, 2010) 关于偶表面积测度的Orlicz Minkowski问题的推广。

关键词：弦积分；弦测度；Minkowski问题

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