Wuhan University
Journal of Natural Sciences

2025, Vol.30 No.6, 535-539

Article ID 1007-1202(2025)06-0535-05 DOI https://doi.org/10.1051/wujns/2025306535

Cite this article: LIN Youjiang, XIAO Qingqing. The $L_{p,s}$-Gaussian-Minkowski Problem on Even Measures [J]. *Wuhan Univ J of Nat Sci*, 2025, 30(6): 535-539.

The $L_{p,s}$-Gaussian-Minkowski Problem on Even Measures

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Abstract: In this paper, we introduce the concept of the $L_{p,s}$-Gaussian surface area measure of a convex body in n-dimensional Euclidean space $\mathbb{R}^n$ and formulate the corresponding $L_{p,s}$-Gaussian-Minkowski problem: Given a finite Borel measure μ on S^{n-1}, what are the necessary and sufficient conditions for the existence of a convex body whose $L_{p,s}$-Gaussian surface area measure equals measure μ? Furthermore, we present a solution to the $L_{p,s}$-Gaussian-Minkowski problem for the case of even measures.

Key words: convex geometry analysis; L_p-Minkowski problem; Gaussian-Minkowski problem; s-Gauss measure; variational equation

CLC number: O186.5

0 Introduction

In 1897, Minkowski posed the Minkowski problem[1]. Minkowski[1-2] resolved its discrete 3D case. For arbitrary measures, Aleksandrov[3], Fenchel and Jessen[4] established complete solutions via variational methods, requiring the convex body's centroid to lie at the origin.

Lutwak[5] established the L_p-Brunn-Minkowski theory, formulating the L_p Minkowski problem[6-9]. Further generalization emerges through Orlicz-Brunn-Minkowski theory[10-14], extending the L_p framework[15].

Recent studies have extended Minkowski problems to Gaussian spaces. Huang *et al*[16] introduced the Gaussian surface area measure $S_{\gamma_n,K}$, establishing the Gaussian-Minkowski problem with existence and uniqueness in classical cases. Feng *et al*[17] proposed L_p Gaussian surface area measures and corresponding Minkowski problems, proving existence in normalized settings.

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. Let n denote a vector in $\mathbb{R}^n$. The s-Gaussian probability measure γ_n^s for $E\subset\mathbb{R}^n$ and $s\in(0,\infty)$ is outlined in Ref. [18] as follows:

$$\gamma_n^s(E)=\frac{1}{C_{s,n}}\int_E\left(1-\frac{s|\boldsymbol{x}|^2}{2}\right)_+^{\frac{1}{s}}\mathrm{d}\boldsymbol{x},\tag{1}$$

where $C_{s,n}:=\frac{n\omega_n}{2}\sqrt{\frac{2}{s}}^{\,n}\beta(\frac{1}{s}+1,\ \frac{n}{2})$, and ω_n is the volume of the unit ball. When $s\to0^+$, γ_n^s is the classical Gaussian probability measure

$$\gamma_n(E)=\frac{1}{(\sqrt{2\pi})^n}\int_E\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{2}}\mathrm{d}\boldsymbol{x}.$$

The generalized Gaussian probability space[19-20] features the generalized Gaussian density $g_{\alpha,q}$. Ref. [21] introduced a generalized Gaussian surface area measure. Moreover, Liu *et al*[21] proposed the generalized Gaussian-Minkowski problem and established the existence and uniqueness of solutions in the smooth case.

Definition 1 Let $p>1$, $s>0$ and $K\in\mathcal{K}_o^n$. The $L_{p,s}$-

Received date: 2025-03-01 © Wuhan University 2025

Foundation item: Supported by the National Natural Science Foundation of China (11971080, 12371137)

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Gaussian surface area measure of K, denoted by $S_{\gamma_n^s,p,K}$, is a Borel measure on S^{n-1} given by

$$S_{\gamma_n^s,p,K}(\eta)=\frac{1}{pC_{s,n}}\int_{\boldsymbol{v}_K^{-1}(\eta)}\left(\boldsymbol{x}\cdot\boldsymbol{v}_K(\boldsymbol{x})\right)^{1-p}\left(1-\frac{s|\boldsymbol{x}|^2}{2}\right)_+^{\frac{1}{s}}\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x}) \quad (2)$$

for any Borel measurable $\eta\subset S^{n-1}$. When $s\to 0^+$, then the $L_{p,s}$-Gaussian surface area measure is the L_p-Gaussian surface area measure.

The $L_{p,s}$-Gaussian-Minkowski Problem: If $p>1$, given a finite Borel measure μ on the S^{n-1}, what is the sufficient and necessary condition on the measure μ so that there exists a convex body $K\in\mathcal{K}_o^n$ and $\alpha\in(0,\frac{1}{n})$ such that $\mu=\frac{S_{\gamma_n^s,p,K}}{\gamma_n^s(K)^{1-\alpha}}$.

The following theorem present a solution to the $L_{p,s}$-Gaussian-Minkowski problem for the case of even measures.

Theorem 1 Suppose μ is a non-zero finite even Borel measure not concentrated in any closed hemisphere and $\alpha\in(0,\frac{1}{n})$. Then there exists an o-symmetric convex body $K\in\mathcal{K}_e^n$ for $s\in(0,\infty)$ such that $\mu=\frac{S_{\gamma_n^s,p,K_s}}{\gamma_n^s(K_s)^{1-\alpha}}$.

1 Preliminaries

We introduce some notations and, for ease of reference, summarize some basic properties of convex bodies. For comprehensive treatments of convex body theory, we refer readers to the texts by Gardner[22], Gruber[23], and Schneider[24], etc.

Let $\boldsymbol{x}\in\mathbb{R}^n$, the norm of a vector $\boldsymbol{x}$ is given by $|\boldsymbol{x}|=\sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$, $\bar{\boldsymbol{x}}$ denotes $\frac{\boldsymbol{x}}{|\boldsymbol{x}|}$. The unit sphere in $\mathbb{R}^n$ is denoted by S^{n-1}. Let $C(S^{n-1})$ represent the set of continuous functions on S^{n-1}. Furthermore, $C_e^+(S^{n-1})$ denotes the set of even, positive, continuous functions on S^{n-1}. For a finite measure μ on S^{n-1}, the total mass of μ is denoted by $|\mu| = \mu(S^{n-1})$.

1.1 Convex Bodies

The set of convex bodies (compact convex subsets) in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$, the set of convex bodies containing the origin in their interiors is denoted by $\mathcal{K}_o^n$, and $\mathcal{K}_e^n\subset\mathcal{K}_o^n$ denotes the set of origin-symmetric convex bodies. The unit ball centered at the origin in $\mathbb{R}^n$ is denoted by B^n, and its volume by ω_n. ∂K denotes the boundary of $K\in\mathcal{K}^n$.

The support function, $h_K: S^{n-1}\to\mathbb{R}$, of a nonempty compact convex set K, is the continuous function on the unit sphere S^{n-1}, defined by $h_K(\boldsymbol{u})=\max\{\boldsymbol{u}\cdot\boldsymbol{x}: \boldsymbol{x}\in K\}$. The radial function, $\rho_K: S^{n-1}\to\mathbb{R}$ for K is the continuous function, defined by $\rho_K(\boldsymbol{x})=\max\{\lambda: \lambda\boldsymbol{x}\in K\}$.

For $K\in\mathcal{K}_o^n$, the polar body K^* of K is defined by $K^*=\{\boldsymbol{x}\in\mathbb{R}^n: \boldsymbol{y}\in K,\ \boldsymbol{x}\cdot\boldsymbol{y}\leqslant 1\}$.

Let $h\in C^+(S^{n-1})$, the Wulff shape $[h]$ associated with h be defined as $[h]=\{\boldsymbol{x}\in\mathbb{R}^n: \boldsymbol{x}\cdot\boldsymbol{v}\leqslant h(\boldsymbol{v}),\ \boldsymbol{v}\in S^{n-1}\}$. The collection of nonempty compact convex sets can be viewed as a metric space with the Hausdroff metric, where the Hausdorff distance between K and L is defined by

$$d(K,L)=\min\{t\geqslant 0: K\subset L+tB^n, L\subset K+tB^n\}.$$

1.2 The Radial Gauss Map of a Convex Body

For $K\in\mathcal{K}_o^n$ and each $\boldsymbol{v}\in S^{n-1}$, the hyperplane

$$H_K(\boldsymbol{v})=\{\boldsymbol{x}\in\mathbb{R}^n: \boldsymbol{x}\cdot\boldsymbol{v}=h_K(\boldsymbol{v})\}$$

is called the supporting hyperplane to K with unit normal $\boldsymbol{v}$. For $\sigma\subset\partial K$, the Gaussian image of σ is defined by

$$\boldsymbol{v}_K(\sigma)=\{\boldsymbol{v}\in S^{n-1}: \ \boldsymbol{x}\in H_K(\boldsymbol{v}) \text{ for some } \boldsymbol{x}\in\sigma\}\subset S^{n-1}.$$

For $\eta\subset S^{n-1}$, the reverse Gaussian image of η is defined by $\boldsymbol{v}_K^{-1}(\eta)=\{\boldsymbol{x}\in\partial K: \boldsymbol{x}\in H_K(\boldsymbol{v}) \text{ for some } \boldsymbol{v}\in\eta\}\subset\partial K$.

Huang *et al*[25] gave the definitions of radial Gaussian image. For $K\in\mathcal{K}_o^n$ and $\eta\subset S^{n-1}$, the radial Gaussian image $\boldsymbol{\alpha}_K(\boldsymbol{u})$ is the set of outer unit normals of K from the boundary points $\rho_K(\boldsymbol{u})\boldsymbol{u}$, for some $\boldsymbol{u}\in\eta$; that is,

$$\boldsymbol{\alpha}_K(\eta)=\bigcup_{\boldsymbol{u}\in\eta}\{\boldsymbol{v}\in S^{n-1}: \rho_K(\boldsymbol{u})\boldsymbol{u}\cdot\boldsymbol{v}=h_K(\boldsymbol{v})\}.$$

Denote $\omega_K\subset S^{n-1}$ as the set of $\boldsymbol{u}\in S^{n-1}$ such that $\boldsymbol{\alpha}_K(\boldsymbol{u})$ contains more than one point; that is, the point $\rho_K(\boldsymbol{u})\boldsymbol{u}\in\partial K$ has more than one outer unit normal. We now define the radial Gauss map $\alpha_K: S^{n-1}\backslash\omega_K\to S^{n-1}$, satisfying $\boldsymbol{\alpha}_K(\{\boldsymbol{u}\})=\{\alpha_K(\boldsymbol{u})\}$.

2 Variational Formulas for the $L_{p,s}$-Gaussian Surface Area Measure

In this section, we derive the variational formula for the $L_{p,s}$-Gaussian surface area measure.

Lemma 1 (Ref.[16], Lemma 3.2) Let $K\in\mathcal{K}_o^n$ and $g\in C(S^{n-1})$. Suppose $\delta>0$ is sufficiently small so that for each $t\in(-\delta,\delta)$, there is $h_t=h_K+tg>0$. Then,

$$\lim_{t\to 0}\frac{\rho_{[h_t]}(\boldsymbol{u})-\rho_K(\boldsymbol{u})}{t}=\frac{g(\alpha_K(\boldsymbol{u}))}{h_K(\alpha_K(\boldsymbol{u}))}\rho_K(\boldsymbol{u}),$$

for almost all $\boldsymbol{u}\in S^{n-1}$ with respect to spherical Lebesgue measure. Moreover, there exists a constant $M>0$, such that

$$\left|\rho_{[h_t]}(\boldsymbol{u})-\rho_K(\boldsymbol{u})\right|<M|t| \tag{3}$$

for all $\boldsymbol{u}\in S^{n-1}$ and $t\in(-\delta,\delta)$.

Theorem 2 Let $K\in\mathcal{K}_o^n$, $s\in(0,\infty)$, $g\in C(S^{n-1})$. Then,

$$\lim_{t\to 0}\frac{\gamma_n^s([\left(h_K^p+tg\right)^{\frac{1}{p}}])-\gamma_n^s(K)}{t}=\int_{S^{n-1}} g\,\mathrm{d}S_{\gamma_n^s,p,K}.$$

Proof Let $h_t=\left(h_K^p+tg\right)^{\frac{1}{p}}$, then

$$h_t=\left(h_K^p+tg\right)^{\frac{1}{p}}=h_K+\frac{1}{p}h_K^{p(\frac{1}{p}-1)}gt+o(t)$$
$$=h_K+\frac{1}{p}h_K^{1-p}gt+o(t), \tag{4}$$

where $o(t)$ is the same order infinitesimal of t. It is evident that $[h_t]\in\mathcal{K}_o^n$ when t is sufficiently small. By the definition of the s-Gaussian probability measure in (1) and employing polar coordinates, we obtain the following expression:

$$\gamma_n^s\left(\left[h_t\right]\right)=\frac{1}{C_{s,n}}\int_{[h_t]}\left(1-\frac{s|\boldsymbol{x}|^2}{2}\right)_+^{\frac{1}{s}}\mathrm{d}\boldsymbol{x}$$
$$=\frac{1}{C_{s,n}}\int_{S^{n-1}}\int_0^{\rho_{[h_t]}(\boldsymbol{u})}\left(1-\frac{s|r\boldsymbol{u}|^2}{2}\right)_+^{\frac{1}{s}}r^{n-1}\,\mathrm{d}r\mathrm{d}\boldsymbol{u}. \tag{5}$$

Denote $F(t)=\int_0^t\left(1-\frac{s|r\boldsymbol{u}|^2}{2}\right)_+^{\frac{1}{s}}r^{n-1}\,\mathrm{d}r$. By the mean value theorem and (3), we have

$$\left|F(\rho_{[h_t]}(\boldsymbol{u}))-F(\rho_K(\boldsymbol{u}))\right|=\left|F'(\theta)\right|\left|\rho_{[h_t]}(\boldsymbol{u})-\rho_K(\boldsymbol{u})\right|$$
$$\leqslant M\left|F'(\theta)\right||t|,$$

where θ is between $\rho_{[h_t]}(\boldsymbol{u})$ and $\rho_K(\boldsymbol{u})$. Therefore, by definition of F, we have $|F'(\theta)|$ is bounded from above by some constant M_1. Furthermore, there exists $M_2=M_1M>0$ such that

$$|F(\rho_{[h_t]}(\boldsymbol{u}))-F(\rho(\boldsymbol{u}))|\leqslant M_2\,t.$$

Together with (5), the dominated convergence theorem, Lemma 1, (4) and (2), we attain

$$\lim_{t\to 0}\frac{\gamma_n^s([\left(h_K^p+tg\right)^{\frac{1}{p}}])-\gamma_n^s(K)}{t}$$
$$=\lim_{t\to 0}\frac{1}{C_{s,n}}\int_{S^{n-1}}\frac{1}{t}\int_{\rho_K(\boldsymbol{u})}^{\rho_{[h_t]}(\boldsymbol{u})}\left(1-\frac{s|r\boldsymbol{u}|^2}{2}\right)_+^{\frac{1}{s}}r^{n-1}\,\mathrm{d}r\mathrm{d}\boldsymbol{u}$$
$$=\frac{1}{C_{s,n}}\int_{S^{n-1}}\frac{1}{p}\,h_K^{1-p}(\boldsymbol{u})g(\alpha_K(\boldsymbol{u}))\left(1-\frac{s\rho_K(\boldsymbol{u})^2}{2}\right)_+^{\frac{1}{s}}\frac{\rho_K(\boldsymbol{u})^n}{h_K(\alpha_K(\boldsymbol{u}))}\,\mathrm{d}\boldsymbol{u}$$
$$=\frac{1}{C_{s,n}}\int_{\partial' K}\frac{1}{p}\,(\boldsymbol{x}\cdot\boldsymbol{v}_K(\boldsymbol{x}))^{1-p}g\left(\boldsymbol{v}_K(\boldsymbol{x})\right)\left(1-\frac{s|\boldsymbol{x}|^2}{2}\right)_+^{\frac{1}{s}}\mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x})$$
$$=\int_{S^{n-1}}g(\boldsymbol{u})\,\mathrm{d}S_{\gamma_n^s,p,K}(\boldsymbol{u}),$$

which is the desired conclusion.

3 The $L_{p,s}$-Gaussian-Minkowski Problem and Its Solution

In this section, we focus on the normalized version of the $L_{p,s}$-Gaussian-Minkowski problem for even measure when $s\in(0,\infty)$. For $0<\alpha<\frac{1}{n}$ and $p>1$, we first transform the problem of the existence of solutions to the $L_{p,s}$-Gaussian-Minkowski problem into the following maximisation problem:

$$\sup\{F_s(f): f\in C_e^+(S^{n-1})\} \tag{6}$$

where F_s: $C_e^+\left(S^{n-1}\right)\to\mathbb{R}$ is given for $f\in C_e^+\left(S^{n-1}\right)$ by

$$F_s(f)=\frac{1}{\alpha}\gamma_n^s\left(\left[f\right]\right)^{\alpha}-\int_{S^{n-1}}f^p\,\mathrm{d}\mu. \tag{7}$$

For $K\in\mathcal{K}_e^n$, denote $F_s(K)=F_s(h_K)$.

Lemma 2 Let $0<\alpha<\frac{1}{n}$ and $p>1$. If an even function f_s is a maximizer to the optimization problem (6), then f_s must be the support function of an o-symmetric convex body; that is, there exists an o-symmetric convex body $K_s\in\mathcal{K}_e^n$ such that $f_s=h_{K_s}$. Moreover, K_s satisfies

$$\mu=\frac{S_{\gamma_n^s,p,K_s}}{\gamma_n^s(K_s)^{1-\alpha}}. \tag{8}$$

Proof Note that for each $f\in C_e^+(S^{n-1})$, $h_{[f]}\leqslant f$ and the Wulff shape of $h_{[f]}$ is the same as that of f. So according to (7), we have $F_s(f)\leqslant F_s(h_{[f]})$.

Therefore, the maximizer of (6) f_s must be the support function of some o-symmetric convex body $K_s\in\mathcal{K}_e^n$. We now use the variational formula to establish that K_s satisfies (8). To this end, for each $g\in C_e(S^{n-1})$, consider the family of Wulff shapes

$$K_t=\left[\left(h_{K_s}^p+tg\right)^{\frac{1}{p}}\right]\in\mathcal{K}_e^n,$$

for t small enough.

Since $f_s=h_{K_s}$ is a maximizer of (6) and the definition of F_s given in equation (7) and by Theorem 2, it follows that

$$0=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}F_s(h_{K_t})=\gamma_n^s(K_s)^{\alpha-1}\int_{S^{n-1}}g\,\mathrm{d}S_{\gamma_n^s,p,K_s}-\int_{S^{n-1}}g\,\mathrm{d}\mu.$$

Note that the above equation holds for every $g\in C_e(S^{n-1})$. Therefore, we conclude that K_s satisfies (8).

Lemma 3 Let $0<\alpha<\frac{1}{n}$, $p>1$ and $s\in(0,\infty)$. For sufficiently small $r>0$, we have $F_s(rB^n)>0$.

Proof From the definition of F_s given in equation

(7), we have

$$F_s(rB^n)=\frac{1}{\alpha}\left(\frac{1}{C_{s,n}}\int_{rB^n}\left(1-\frac{s|\boldsymbol{x}|^2}{2}\right)_+^{\frac{1}{s}}\mathrm{d}\boldsymbol{x}\right)^{\alpha}-\int_{S^{n-1}}r^p\,\mathrm{d}\mu$$

$$=\frac{1}{\alpha}\left(\frac{n\omega_n}{C_{s,n}}\int_0^r\left(1-\frac{st^2}{2}\right)_+^{\frac{1}{s}}t^{n-1}\,\mathrm{d}t\right)^{\alpha}-r^p|\mu|$$

$$=r\left[\frac{1}{\alpha}\left(\frac{n\omega_n}{C_{s,n}}\right)^{\alpha}\left(\frac{\int_0^r\left(1-\frac{st^2}{2}\right)_+^{\frac{1}{s}}t^{n-1}\,\mathrm{d}t}{r^{\frac{1}{\alpha}}}\right)^{\alpha}-r^{p-1}|\mu|\right]. \quad (9)$$

It follows from L'Hôpital's rule[26] that when $0<\alpha<\frac{1}{n}$, we have

$$\lim_{r\to0^+}\frac{\int_0^r\left(1-\frac{st^2}{2}\right)_+^{\frac{1}{s}}t^{n-1}\,\mathrm{d}t}{r^{\frac{1}{\alpha}}}=\lim_{r\to0^+}\frac{\alpha\left(1-\frac{st^2}{2}\right)_+^{\frac{1}{s}}r^{n-1}}{r^{\frac{1}{\alpha}-1}}$$

$$=\alpha\lim_{r\to0^+}r^{n-\frac{1}{\alpha}}=\infty. \quad (10)$$

According to (9), (10) and $p>1$, for r is sufficiently small, $F_s(rB^n)>0$.

Proof of Theorem 1 By Lemma 2, it suffices to show that a maximizer to the optimization problem (6) exists. We assume that $\{K_i\}_{i=1}^{\infty}$ is a sequence of o-symmetric convex bodies and

$$\lim_{i\to\infty}F_s(K_i)=\sup\{F_s(f): f\in C_e^+(S^{n-1})\}\geqslant F_s(rB^n)>0.$$

Choose $r_i>0$ and $\boldsymbol{u}_i\in S^{n-1}$ such that $r_i\boldsymbol{u}_i\in K_i$ and $r_i=\max_{\boldsymbol{u}\in S^{n-1}}\rho_{K_i}(\boldsymbol{u})$.

It is simple to notice that $K_i\subset r_iB^n$. We claim that r_i is a bounded sequence from above with upper bound M. Otherwise, by taking a subsequence r_{i_j}, we may assume that $\lim_{j\to\infty}r_{i_j}=+\infty$. Since $K_{i_j}\subset r_{i_j}B^n$ and (7), we have

$$F_s(K_{i_j})\leqslant\frac{1}{\alpha}\gamma_n^s(r_{i_j}B^n)^{\alpha}-\int_{S^{n-1}}h_{K_{i_j}}^p\,\mathrm{d}\mu.$$

Since $r_{i_j}\boldsymbol{u}_{i_j}\in K_{i_j}$, we have $h_{K_{i_j}}(\boldsymbol{v})\geqslant r_{i_j}\boldsymbol{u}_{i_j}\cdot\boldsymbol{v}$. Therefore, we have

$$F_s(K_{i_j})\leqslant\frac{1}{\alpha}\gamma_n^s(r_{i_j}B^n)^{\alpha}-r_{i_j}^p\int_{\{\boldsymbol{v}\in S^{n-1}:\boldsymbol{v}\cdot\boldsymbol{u}_{i_j}\geqslant0\}}|\boldsymbol{v}\cdot\boldsymbol{u}_{i_j}|^p\mathrm{d}\mu. \quad (11)$$

By the fact that μ is even and not concentrated in any closed hemisphere, there exists a constant $c_0>0$ such that

$$\int_{\{\boldsymbol{v}\in S^{n-1}:\boldsymbol{v}\cdot\boldsymbol{u}_{i_j}\geqslant0\}}|\boldsymbol{v}\cdot\boldsymbol{u}_{i_j}|^p\mathrm{d}\mu\geqslant c_0 \quad (12)$$

as $r_{i_j}\to+\infty$, $\gamma_n^s(r_{i_j}B^n)\to\gamma_n^s(\mathbb{R}^n)=1$. Therefore, according to (11), (12),

$$F_s(K_{i_j})\leqslant\frac{1}{\alpha}\gamma_n^s(r_{i_j}B^n)^{\alpha}-r_{i_j}^p c_0^p<0$$

as $j\to\infty$. The above equation is in contradiction to

$$\lim_{j\to\infty}F_s(K_{i_j})=\sup\{F_s(f): f\in C_e^+(S^{n-1})\}>0.$$

This leads to a contradiction. Therefore, the sequence of convex bodies $\{K_i\}_{i=1}^{\infty}$ is uniformly bounded.

We use Blaschke selection theorem and assume (by taking a subsequence) that $\{K_i\}_{i=1}^{\infty}$ converges to some $K_0\in\mathcal{K}_e^n$ in Hausdorff metric. Note that by the continuity of the s-Gaussian volume with respect to the Hausdorff metric, definition of F_s, and $F_s(rB^n)>0$, we have

$$\frac{1}{\alpha}\gamma_n^s(K_0)^{\alpha}\geqslant F_s(K_0)=\lim_{i\to\infty}F_s(K_i)>0.$$

Combining the inequality mentioned above, with the fact that K_0 is o-symmetric, implies that K_0 contains the origin in its interior. Therefore, the convex body K_0 is a maximizer to the optimization (6).

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偶测度的$L_{p,s}$-高斯-Minkowski问题

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摘要：本文引入了n维欧氏空间$\mathbb{R}^n$中凸体的$L_{p,s}$-高斯表面积测度的概念， 并提出了相应的$L_{p,s}$-高斯-Minkowski问题， 即当一个S^{n-1}上的有限Borel测度μ满足什么充要条件时保证存在一个凸体， 使得该凸体的$L_{p,s}$-高斯表面积测度等于测度μ。 此外， 我们给出了关于偶测度的$L_{p,s}$-高斯-Minkowski问题的一个解。

关键词：凸几何分析；L_p-Minkowski问题；高斯-Minkowski问题；s-高斯测度；变分公式

□