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Fixed Point Results for Extended Contractions with Various Weak Conditions in Generalized Metric Spaces

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Abstract: In this work, we investigate several fixed point theorems in double controlled metric spaces by providing some new definitions, which serve as a natural extension of the existing notions. Moreover, we establish its various characterizations by some typical examples. Specifically, these examples hold in double-controlled metric-like spaces instead of usual *b*-metric spaces. The fixed point theorems are obtained under these new concepts, without appealing to the completeness of the spaces or the continuity of the mappings. Furthermore, we prove that the completeness in double-controlled metric-like spaces is necessary if the extended Kannan-type contractions have a fixed point in the spaces. The results contribute to a deeper understanding of Kannan-type contractions and their applicability in the context of fixed point theory in double-controlled metric spaces.

Key words: double controlled metric-like spaces; completeness; asymptotic regularity; fixed point

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0 Introduction

Fixed point results are widely used for finding solutions to various differential equations and integral equations. Since the valuable Banach contraction principle was created by Polish mathematician Banach in 1922[1], many fixed point theorems about different contractions were discussed, such as the Kannan-type fixed point theorem[2]. There are lots of other important contractions which were given by Reich[3-4], Ciric[5], Chatterjea[6], Hardy and Rogers[7], Bianchini[8], and so on. Among them, the Kannan-type contraction gets more attention, since the Kannan-type fixed point theorems describe the relationship between the contraction and the completeness of metric spaces[9].

Some other more generalized spaces were given by many researchers, which extended the structure of metric spaces, such as *b*-metric spaces[10-11] and extended *b*-metric spaces[12]. Numerous fixed point theorems were proved on these spaces. Whereafter, the notion of controlled metric-type space was introduced in Ref. [13], as a generation of extended *b*-metric space. In 2020, Mlaiki *et al*[14] initiated a double double-controlled metric-type space. Later, a more general concept named double-controlled metric-like space appeared in Refs. [15-16], which was a sharp extension of all the types of metric

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spaces mentioned above. The authors established some fixed point results in this space.

Recently, Górnicki[17] and Garai *et al*[18] obtained some meaningful results for Kannan-type contractive self-mappings in metric spaces. In this paper, we prove some fixed point theorems about extended Kannan contraction in double controlled metric-like spaces, by introducing the concepts of bounded compactness, orbital compactness, weak orbital continuity, orbital completeness and asymptotic regularity in double controlled metric-like spaces. As compared to the existing results from Refs. [15,17-19], our main results weaken the compactness and completeness of the spaces, the orbital continuity and the k-continuity of the mappings. Furthermore, we prove that the completeness in double controlled metric-like spaces is necessary if the extended Kannan-type contractions have a fixed point in the set M. Besides these progresses, we also give some examples to illustrate that our new notions and theorems are genuine improvements and generalizations of the corresponding results in the literature.

1 Preliminaries

First, we recall some necessary definitions in double-controlled metric-like spaces.

Definition 1[15] Assume a set $M\neq\varnothing$ and $\alpha,\beta: M\times M\to[1,+\infty)$ be two noncomparable functions. The function $\zeta: M\times M\to[0,+\infty)$ satisfies for all $f, g, h\in M$:
1) $\zeta(f,g)=0\Rightarrow f=g$,
2) $\zeta(f,g)=\zeta(g,f)$,
3) $\zeta(f,g)\leqslant\alpha(f,h)\zeta(f,h)+\beta(h,g)\zeta(h,g)$.

In this case, (M,ζ) is called a double controlled metric-like space (DCMLS).

Note that if the condition 1) is replaced by $\zeta(f,g)=0$ if and only if $f=g$, then it is a double controlled metric-type space (DCMTS)[14]. Moreover, if the triangle inequality is $\zeta(f,g)\leqslant\alpha(f,h)\zeta(f,h)+\alpha(h,g)\zeta(h,g)$, then it is a controlled metric-type space (CMTS)[13]. Next, if this inequality is further written as $\zeta(f,g)\leqslant\alpha(f,g)[\zeta(f,h)+\zeta(h,g)]$, then it is an extended b-metric space (EbMS)[12]. Therefore, a double controlled metric-like space must be a double controlled metric-type space, a controlled metric-type space, an extended b-metric space, a b-metric space and a metric space, but not the converse. The following example greatly explores their relationships.

Example 1 Let $M=\mathbb{R}^+$ and $\zeta: M\times M\to\mathbb{R}^+$ be defined as

$$\zeta(f,g)=\begin{cases}0\Rightarrow & f=g,\\ \dfrac{1}{4}, & \text{if } f=g=0,\\ \dfrac{1}{f+1}, & \text{if } f\geqslant 1 \text{ and } g\in[0,1),\\ \dfrac{1}{g+1}, & \text{if } g\geqslant 1 \text{ and } f\in[0,1),\\ 1, & \text{otherwise.}\end{cases}$$

The functions $\alpha,\beta: M\times M\to[1,+\infty)$ are considered as

$$\alpha(f,g)=\begin{cases}1, & f,g\geqslant 1,\\ f+1, & \text{otherwise,}\end{cases}$$

and $\beta(f,g)=\begin{cases}1, & f,g<1,\\ 2(f+g), & \text{otherwise.}\end{cases}$

Obviously, 1) and 2) of Definition 1 are satisfied. Let's prove that condition 3) is also satisfied.

Case 1 If $f\geqslant 1,\ 0\leqslant g<1$, then

$$\zeta(f,g)=\frac{1}{f+1}\leqslant\begin{cases}1\times 1+2(h+g)\cdot\dfrac{1}{h+1}, & h\geqslant 1,\\ (f+1)\cdot\dfrac{1}{f+1}+1\times 1, & 0\leqslant h<1.\end{cases}$$

Case 2 If $0\leqslant f<1, g\geqslant 1$, then

$$\zeta(f,g)=\frac{1}{g+1}\leqslant\begin{cases}(f+1)\cdot\dfrac{1}{h+1}+2(h+g)\cdot 1, & h\geqslant 1,\\ (f+1)\cdot 1+2(h+g)\cdot\dfrac{1}{g+1}, & 0\leqslant h<1.\end{cases}$$

Case 3 If $f,g\geqslant 1$, then

$$\zeta(f,g)=1\leqslant\begin{cases}1\times 1+2(h+g)\cdot 1, & h\geqslant 1,\\ (f+1)\cdot\dfrac{1}{f+1}+2(h+g)\cdot\dfrac{1}{g+1}, & 0\leqslant h<1.\end{cases}$$

Case 4 If $f,g<1$, $f\neq 0$ or $g\neq 0$, then

$$\zeta(f,g)=1\leqslant\begin{cases}(f+1)\cdot\dfrac{1}{h+1}+2(h+g)\cdot\dfrac{1}{h+1}, & h\geqslant 1,\\ (f+1)\cdot 1+1\times 1, & 0\leqslant h<1.\end{cases}$$

Case 5 If $f=g=0$, then

$$\zeta(f,g)=\frac{1}{4}\leqslant(f+1)\cdot\frac{1}{4}+1\times\frac{1}{4},\quad h=0,$$

if $h\neq 0$, it also holds.

Thus, the condition 3) of Definition 1 is satisfied in the five cases. Therefore, (M,ζ) is a DCMLS. But, $\zeta(0,0)\neq 0$ shows that (M,ζ) is not a DCMTS. Furthermore, (M,ζ) is not an extended b-metric space or a controlled metric-type space.

Definition 2[15] Let (M,ζ) be a DCMLS and $\{f_n\}$ be a sequence in M. Then,

(i) $\{f_n\}$ converges to f (denoted $f_n\to f$) if and only if

$$\lim_{n\to\infty}\zeta(f_n,f)=\zeta(f,f);$$

(ii) $\{f_n\}$ is a ζ-Cauchy if and only if

$$\lim_{n,m\to\infty}\zeta(f_n,f_m) \text{ exists and is finite;}$$

(iii) (M,ζ) is a complete space if for every ζ-Cauchy sequence $\{f_n\}$, there exists $f\in M$ such that

$$\lim_{n\to\infty}\zeta(f_n,f)=\zeta(f,f)=\lim_{n,m\to\infty}\zeta(f_n,f_m).$$

2 Bounded Compactness and τ-Orbital Compactness

The concepts of bounded compactness and τ-orbital compactness were studied in usual metric spaces in Ref. [18], which were significant in weakening the condition of compactness. Moreover, Jaggi[20] and Pant *et al*[21] gave the notions of f_0-orbital continuity and weak orbital continuity in metric spaces, respectively, which generalized the concepts of orbital continuity and k-continuity. In the following, we introduce the notions of extended Kannan-type contraction, bounded compactness, τ-orbital compactness, f_0-orbital continuity and weak orbital continuity in the framework of DCMLS. Of course, they are true in controlled metric spaces and extended b-metric spaces, etc.

Definition 3 Let (M,ζ) be a DCMLS. The mapping $\tau: M\to M$ is said to be an extended Kannan-type contraction if it satisfies

$$\zeta(\tau f,\tau g)<a\zeta(f,\tau f)+b\zeta(g,\tau g), \tag{1}$$

for all $f,g\in M$ with $f\neq g$, where $a+b=1$ and $a,b\in(0,1)$.

Definition 4 Let (M,ζ) be a DCMLS and τ be a self-mapping on M. Let $f\in M$ and $O_\tau(f)=\{f,\tau f,\tau^2 f,\tau^3 f,\cdots\}$.

1) The space (M,ζ) is said to be boundedly compact if every bounded sequence in M has a convergent subsequence.

2) The set M is said to be τ-orbitally compact set if every sequence in $O_\tau(f)$ has a convergent subsequence for all $f\in M$.

3) The mapping τ is said to be f_0-orbitally continuous for some $f_0\in M$ if its restriction to the set $\overline{O(\tau,f_0)}$ is continuous, i. e., $\tau:\overline{O(\tau,f_0)}\to M$ is continuous; here, $\overline{O(\tau,f_0)}$ represents the closure of $O(\tau,f_0)$. Moreover, τ is said to be orbitally continuous if it is f_0-orbitally continuous for all $f_0\in M$.

4) The mapping τ is said to be weakly orbitally continuous if the set

$$\{g\in M: u=\lim_{i\to\infty}\tau^{n_i}g \text{ implies } \tau u=\lim_{i\to\infty}\tau\tau^{n_i}g\}$$

is nonempty whenever the set $\{f\in M: u=\lim_{i\to\infty}\tau^{n_i}f\}$ is nonempty for $u\in M$.

Given the above definitions, compactness of a set implies bounded compactness and τ-orbital compactness of this set, but not the converse. In the same way, orbital continuity and k-continuity of τ imply weak orbital continuity and f_0-orbital continuity, but the converse need not be true. The next example fully illustrates this point.

Example 2 1) Let $M=\mathbb{R}^+$ and $\zeta: M\times M\to\mathbb{R}^+$ by

$$\zeta(f,g)=\begin{cases}1, & f=g=0,\\ |f-g|, & \text{otherwise.}\end{cases}$$

The functions $\alpha,\beta: M\times M\to[1,\infty)$ are written as

$$\alpha(f,g)=\begin{cases}\dfrac{1}{f}, & g=0, f\in(0,1],\\ \dfrac{1}{g}, & f=0, g\in(0,1],\\ \max\{\dfrac{1}{f},\dfrac{1}{g}\}, & f,g\in(0,1],\\ 1, & \text{otherwise,}\end{cases}$$

and $\beta(f,g)=\begin{cases}\dfrac{1}{f}+\dfrac{1}{g}, & f,g\in(0,1],\\ 1, & \text{otherwise.}\end{cases}$

Then, (M,ζ) is a DCMLS by the detailed calculations. Define the mappings $\tau_1,\tau_2: M\to M$ as

$$\tau_1(f)=\frac{f}{(n+1)^2},\ n\leqslant f<n+1, n\in\mathbb{N},$$

$$\tau_2(f)=2f+1,$$

for all $f\in M$. This implies that M is τ_1-orbitally compact and boundedly compact but not τ_2-orbitally compact.

2) Let $M=[0,2)$. The metric ζ and the functions α,β are defined the same as above and $\tau: M\to M$. It is obvious that M is τ-orbitally compact but not complete.

3) Let $M=[0,2)$. The metric ζ and the functions α,β are defined the same as above. The mapping $\tau: M\to M$ is given as

$$\tau(f)=\begin{cases}\dfrac{1+f}{2}, & 0\leqslant f<1,\\ 2, & 1\leqslant f\leqslant 2.\end{cases}$$

It is not difficult to verify that τ is weakly orbitally continuous. If $f=2$, then $\tau^n(2)\to 2$, $\tau(\tau^n(2))\to 2=\tau(2)$. However, τ is not orbitally continuous, since $\tau^n(0)\to 1$, while $\tau(\tau^n(0))\to 1\neq\tau(1)$. Likewise, we can prove that τ is not k-continuous for any integer $k\geqslant 1$, that is,

$$\tau^{k-1}(\tau^n(0))\to 1, \tau^k(\tau^n(0))\to 1\neq\tau(1).$$

Hence, the conditions of k-continuity and orbital continuity are stronger than the weak orbital continuity for the mapping τ.

Theorem 1 Let (M,ζ) be a boundedly compact DCMLS and $\tau\colon M\to M$ be an extended Kannan-type contraction. The mapping τ is f_0-orbitally continuous for some $f_0\in M$ or weakly orbitally continuous, then τ has a unique fixed point $f^*\in M$ and for each $f\in M$ the iterated sequence $\tau^n f$ converges to f^*, i.e., τ is a Picard operator.

Proof For arbitrary $f_0\in M$, set $f_n=\tau f_{n-1}=\tau^n f_0$, $n\geqslant 1$. Suppose $f_n\neq f_{n+1}$ for all $n\in\mathbb{N}$. Indeed, if for some $n\in\mathbb{N}$, $f_n=f_{n+1}=\tau f_n$, then f_n is the fixed point of τ. Denote $u_n=\zeta(f_n,f_{n+1})$ for each $n\in\mathbb{N}$. By (1), we have $u_n=\zeta(f_n,f_{n+1})=\zeta(\tau f_{n-1},\tau f_n)<a\zeta(f_{n-1},f_n)+b\zeta(f_n,f_{n+1})=au_{n-1}+bu_n$. As $a+b=1$ and $a,b\in(0,1)$, we have $(1-b)u_n<au_{n-1}=(1-b)u_{n-1}$, which gives that $u_n<u_{n-1}$ for all $n\in\mathbb{N}$. Repeat this process, we conclude that $0<\cdots<u_n<u_{n-1}<\cdots<u_0=\zeta(f_0,f_1)$.

So, $\lim\limits_{n\to\infty}u_n=u^*$ for some $u^*\geqslant 0$. It follows that $\zeta(f_n,f_m)=\zeta(\tau f_{n-1},\tau f_{m-1})<a\zeta(f_{n-1},\tau f_{n-1})+b\zeta(f_{m-1},\tau f_{m-1})=au_{n-1}+bu_m<(a+b)u_0=u_0$, for all $n,m\in\mathbb{N}$, which shows $\{f_n\}$ is a bounded sequence in M. By the fact that M is boundedly compact, the sequence $\{f_n\}$ has a convergent subsequence $\{f_{n_i}\}$ and $f^*\in M$ such that $f_{n_i}\to f^*$ as $i\to\infty$.

If τ is f_0-orbitally continuous, then $\tau f_{n_i}\to\tau f^*$. If τ is weakly orbitally continuous, then $\tau^n f_0\to f^*$ for each $f_0\in M$. By weak orbital continuity of τ, we know $\tau^n g_0\to f^*$ and $\tau^{n+1}g_0\to\tau f^*$ for some g_0 in M. Next, let us prove that $f^*=\tau f^*$. Assume $u^*>0$. We have $0<u^*=\lim\limits_{i\to\infty}\zeta(f_{n_i},\tau f_{n_i})=\zeta(f^*,\tau f^*)$.

Moreover, $0<u^*=\lim\limits_{i\to\infty}u_{n_i}=\lim\limits_{i\to\infty}\zeta(\tau f_{n_i},\tau^2 f_{n_i})=\zeta(\tau f^*,\tau^2 f^*)<a\zeta(f^*,\tau f^*)+b\zeta(\tau f^*,\tau^2 f^*)$, which means $u^*=\zeta(\tau f^*,\tau^2 f^*)<\zeta(f^*,\tau f^*)=u^*$, a contradiction. Thus, $u^*=0$ and $f^*=\tau f^*$, i.e., f^* is a fixed point of τ. For the uniqueness of f^*, if $\tau f=f$ for some $f\in M$, then $\zeta(f,f^*)=\zeta(\tau f,\tau f^*)<a\zeta(f,\tau f)+b\zeta(f^*,\tau f^*)=0$, lead to a contradiction.

For τ is a Picard operator, since $a+b=1$ and $a,b\in(0,1)$, we can infer that

$$\begin{aligned}&\zeta(f_{n+1},f^*)=\zeta(\tau f_n,\tau f^*)<a\zeta(f_n,\tau f_n)+b\zeta(f^*,\tau f^*)\\&=a\zeta(f_n,f_{n+1})+b\zeta(f^*,f^*)=a\zeta(f_n,f_{n+1})\to 0(n\to\infty).\end{aligned}$$

Hence, $f_{n+1}=\tau^n f_0\to f^*$ $(n\to\infty)$, i.e., τ is a Picard operator.

Theorem 2 Let (M,ζ) be a τ-orbitally compact DCMLS and $\tau\colon M\to M$ be an extended Kannan-type contraction. The mapping τ is f_0-orbitally continuous for some $f_0\in M$ or weakly orbitally continuous, then τ has a unique fixed point $f^*\in M$ and for each $f\in M$ the iterated sequence $\tau^n f$ converges to f^*, i.e., τ is a Picard operator.

Proof The proof is similar to Theorem 1. We first get the sequence $f_n=\tau f_{n-1}=\tau^n f_0, n\geqslant 1$. Suppose $f_n\neq f_{n+1}$ for all $n\in\mathbb{N}$. Indeed, if for some $n\in\mathbb{N}$, $f_n=f_{n+1}=\tau f_n$, then f_n is the fixed point of τ. Without loss of generality, we assume that $f_n\neq f_{n+1}$ for all $n\in\mathbb{N}$, then $u_n\to u^*\geqslant 0$ $(n\to\infty)$ by using analogous arguments as above. Because M is τ-orbitally compact, there is a convergent subsequence $\{f_{n_i}\}$ of $\{f_n\}$ and $f^*\in M$ such that $f_{n_i}\to f^*$ as $i\to\infty$. By f_0-orbital continuity or weak orbital continuity of τ, we obtain $\tau f_{n_i}\to\tau f^*$. The rest proof is similar to Theorem 1.

Remark 1 The main theorems in our paper are extensions of the results in Refs. [17-18]. The results in these literature always rely strongly on the compactness of the metric spaces. Moreover, Theorem 2.2 in Ref. [17] needs the continuity of the mapping and Theorem 2.1, 2.2 in Ref. [18] need orbital continuity or k-continuity. However, the continuity condition is weakened by weak orbital continuity and the compactness of the spaces is relaxed by bounded compactness and orbital compactness in our results. In addition, we extend $a=b=\frac{1}{2}$ to $a,b\in(0,1)$. As a consequence, we have more conveniences for applications in the future.

From the above two theorems, we immediately obtain the following several assertions.

Corollary 1 Let (M,ζ) be a boundedly compact DCMTS and $\tau{:}M\to M$ be an extended Kannan-type contraction. The mapping τ is f_0-orbitally continuous for some $f_0\in M$ or weakly orbitally continuous, then τ has a unique fixed point $f^*\in M$ and for each $f\in M$ the iterated sequence $\tau^n f$ converges to f^*, i.e., τ is a Picard operator.

Corollary 2 Let (M,ζ) be a τ-orbitally compact DCMTS and $\tau\colon M\to M$ be an extended Kannan-type contraction. The mapping τ is f_0-orbitally continuous for some $f_0\in M$ or weakly orbitally continuous, then τ has a unique fixed point $f^*\in M$ and for each $f\in M$ the iterated sequence $\tau^n f$ converges to f^*, i.e., τ is a Picard operator.

Corollary 3 Let (M,ζ) be a boundedly compact CMTS and $\tau\colon M\to M$ be an extended Kannan-type contraction. The mapping τ is f_0-orbitally continuous for some $f_0\in M$ or weakly orbitally continuous, then τ has a unique fixed point $f^*\in M$ and for each $f\in M$ the iterated sequence $\tau^n f$ converges to f^*, i.e., τ is a Picard operator.

Corollary 4 Let (M,ζ) be a τ-orbitally compact CMTS and $\tau\colon M\to M$ be an extended Kannan-type contraction. The mapping τ is f_0-orbitally continuous for some $f_0 \in M$ or weakly orbitally continuous, then τ has a unique fixed point $f^* \in M$ and for each $f \in M$ the iterated sequence $\tau^n f$ converges to f^*, i.e., τ is a Picard operator.

Corollary 5 Let (M, ζ) be a boundedly compact EbMS and $\tau: M \to M$ be an extended Kannan-type contraction. The mapping τ is f_0-orbitally continuous for some $f_0 \in M$ or weakly orbitally continuous, then τ has a unique fixed point $f^* \in M$ and for each $f \in M$ the iterated sequence $\tau^n f$ converges to f^*, i.e., τ is a Picard operator.

Corollary 6 Let (M, ζ) be a τ-orbitally compact EbMS and $\tau: M \to M$ be an extended Kannan-type contraction. The mapping τ is f_0-orbitally continuous for some $f_0 \in M$ or weakly orbitally continuous, then τ has a unique fixed point $f^* \in M$, and for each $f \in M$, the iterated sequence $\tau^n f$ converges to f^*, i.e., τ is a Picard operator.

There is an example in which the mapping τ has a fixed point since it satisfies our results. However, the space is incomplete and the mapping is not continuous, which means that our results are meaningful.

Example 3 Let $M = (-1, 1] \cup \{2\}$ and define the metric $\zeta: M \times M \to \mathbb{R}^+$ by

$$\zeta(f, g) = \begin{cases} k, & f = g = 0, \\ k|f - g|, & \text{otherwise}' \end{cases}$$

for all $f, g \in M$, where $k > 0$. The functions $\alpha, \beta: M \times M \to [1, +\infty)$ are written as

$$\alpha(f, g) = \begin{cases} \dfrac{1}{|g|} + 1, & f = 0, g \neq 0, \\ 1, & \text{otherwise}, \end{cases}$$

and $\beta(f, g) = \begin{cases} |g| + 1, & f = 0, g \neq 0, \\ 1, & \text{otherwise}. \end{cases}$

Furthermore, define the mapping $\tau: M \to M$ by:

$$\tau f = \begin{cases} \dfrac{f}{5}, & f \neq 1, \\ 0, & f = 1. \end{cases}$$

Obviously, τ is not continuous but f_0-orbitally continuous for $f_0 \in (-1, 1) \cup \{2\}$. Moreover, (M, ζ) is an incomplete but τ-orbitally compact DCMLS. Take $a = \dfrac{1}{4}, b = \dfrac{3}{4}$. There are the following two cases for $f \neq g$.

(i) If $f = 1$ and $g \neq 1$, then

$$\zeta(\tau f, \tau g) = k\left|\frac{g}{5} - 0\right| < \frac{1}{4}k|1 - 0| + \frac{3}{4}k\left|g - \frac{g}{5}\right| = a\zeta(f, \tau f) + b\zeta(g, \tau g).$$

On the other hand, if $f \neq 1$ and $g = 1$, it is obviously true by the symmetry of the double controlled metric.

(ii) If $f, g \in M$ with $f \neq 1, g \neq 1$ and $f \neq g$, then

$$\zeta(\tau f, \tau g) = k\left|\frac{f}{5} - \frac{g}{5}\right| \leqslant k\left|\frac{f}{5}\right| + k\left|\frac{g}{5}\right| < \frac{1}{4}k\left|f - \frac{f}{5}\right| + \frac{3}{4}k\left|g - \frac{g}{5}\right| = a\zeta(f, \tau f) + b\zeta(g, \tau g).$$

Therefore, the mapping τ has a unique fixed point in M by Theorem 1. Since $\tau(0, 0) = k > 0$, the space is not a DCMTS (or CMTS, EbMS etc.). Furthermore, $a = b = \dfrac{1}{2}$ is a special case of $a + b = 1, a, b \in (0, 1)$. Hence, the existing theorems in the literature are not applicable here.

3 Asymptotic Regularity and Orbital Completeness

In the following, we obtain the fixed point theorems of generalized contractive mapping in orbitally complete DCMLS, under the condition of asymptotic regularity. The continuity of the mapping is relaxed. At first, we give the definitions of asymptotic regularity and τ-orbital completeness in DCMLS.

Definition 5 Let (M, ζ) be a DCMLS. The mapping $\tau: M \to M$ is said to be asymptotically regular, if $\lim\limits_{n \to \infty} \zeta(\tau^{n+1} f, \tau^n f) = 0$ for all $f \in M$.

Definition 6 Let (M, ζ) be a DCMLS. The space (M, ζ) is said to be τ-orbitally complete, if every ζ-Cauchy sequence which is contained in $O_\tau(f)$ for some $f \in M$ converges in M. Every complete DCMLS is τ-orbitally complete for any τ, but a τ-orbitally complete DCMLS needs not be complete.

The following theorem is obtained under a generalized version of extended Kannan-type contraction, where $a, b \in [0, +\infty)$.

Theorem 3 Let $\tau: M \to M$ be an asymptotically regular mapping on the τ-orbitally complete DCMLS. The mapping τ is f_0-orbitally continuous for some $f_0 \in M$ or weakly orbitally continuous and satisfies

$$\zeta(\tau f, \tau g) \leqslant a\zeta(f, \tau f) + b\zeta(g, \tau g), \tag{2}$$

for all $f, g \in M$, where $a, b \in [0, +\infty)$. Then, τ has a unique fixed point $f^* \in M$ and for each $f \in M$ the iterated sequence $\tau^n f$ converges to f^*, i.e.,τ is a Picard operator.

Proof For arbitrary $f_0 \in M$, set $f_n = \tau f_{n-1} = \tau^n f_0, n \geqslant 1$. Suppose $f_n \neq f_{n+1}$ for all $n \in \mathbb{N}$. Indeed, if for some $n \in \mathbb{N}, f_n = f_{n+1} = \tau f_n$, then f_n is the fixed point of τ. Thus, we consider that $f_n \neq f_{n+1}$ for any $n \in \mathbb{N}$. For all $m > n$, we have $\zeta(f_n, f_m) = \zeta(\tau f_{n-1}, \tau f_{m-1}) \leqslant a\zeta(f_{n-1}, \tau f_{n-1}) + b\zeta(f_{m-1}, \tau f_{m-1}) = a\zeta(f_{n-1}, f_n) + b\zeta(f_{m-1}, f_m)$.

According to asymptotic regularity, $\lim\limits_{n \to \infty} \zeta(f_{n-1}, f_n) =$

0, so $\{f_n\}$ is a ζ-Cauchy in M. Since (M,ζ) is τ-orbitally complete, there exists $f^*\in M$ such that $f_n\to f^*$ as $n\to\infty$. We shall prove $\tau f^*=f^*$, i.e., f^* is the fixed point of τ.

If τ is f_0-orbitally continuous, then $f_n=\tau_n f_0\to f^*$for each $f_0\in M$.

If τ is weakly orbitally continuous, then $\tau^n f_0\to f^*$ for each $f_0\in M$. By virtue of weak orbital continuity of τ, we know $\tau^n g_0\to f^*$ and $\tau^{n+1}g_0\to\tau f^*$ for some g_0 in M. It implies $f^*=\tau f^*$ since $\tau^{n+1}g_0\to f^*$. f^* is a fixed point of τ.

For the uniqueness of f^*, if $g^*\in M$, $\tau g^*=g^*$, then

$$\zeta(f^*,g^*)=\zeta(\tau f^*,\tau g^*)\leqslant a\zeta(f^*,\tau f^*)+b\zeta(g^*,\tau g^*)=0,$$

which yields $f^*=g^*$. Hence, f^* is the unique fixed point of τ and $f_n=\tau_n f_0\to f^*$ $(n\to\infty)$.

Corollary 7 Let $\tau:M\to M$ be an asymptotically regular mapping on the τ-orbitally complete DCMLS. The mapping τ is f_0 -orbitally continuous for some $f_0\in M$ or weakly orbitally continuous and satisfies

$$\zeta(\tau f,\tau g)\leqslant k[\zeta(f,\tau f)+\zeta(g,\tau g)], \tag{3}$$

for all $f,g\in M$, where $k\in[0,+\infty)$. Then, τ has a unique fixed point $f^*\in M$, and for each $f\in M$, the iterated sequence $\tau^n f$ converges to f^*, i.e., τ is a Picard operator.

Now, if $a\in[0,+\infty), b\in(0,1)$ and the noncomparable functions $\alpha:M\times M\to[1,+\infty)$, $\beta:M\times M\to[1,\frac{1}{b})$, both the limits $\lim\limits_{n\to\infty}\alpha(f,f_n)$ and $\lim\limits_{n\to\infty}\beta(f,f_n)$ exist and are finite for any $f,f_0\in M$ and $f_n=\tau^n f_0, n\geqslant 1$, then the condition "$\tau$ is f_0 -orbitally continuous for some $f_0\in M$ or weakly orbitally continuous" in Theorem 3 can be deleted.

Theorem 4 Let $\tau:M\to M$ be an asymptotically regular mapping on the τ-orbitally complete DCMLS. The mapping τ satisfies

$$\zeta(\tau f,\tau g)\leqslant a\zeta(f,\tau f)+b\zeta(g,\tau g), \tag{4}$$

for all $f,g\in M$, where $a\in[0,+\infty), b\in(0,1)$. If for any $f,f_0\in M, f_n=\tau^n f_0$ and the noncomparable functions $\alpha:M\times M\to[1,+\infty),\beta:M\times M\to[1,\frac{1}{b})$, both the limits $\lim\limits_{n\to\infty}\alpha(f,f_n)$ and $\lim\limits_{n\to\infty}\beta(f,f_n)$ exist and are finite, then τ has a unique fixed point $f^*\in M$, and for each $f\in M$, the iterated sequence $\tau^n f$ converges to f^*, i.e., τ is a Picard operator.

Proof By the proof in the previous part of Theorem 3, we know that there exists $f^*\in M$ such that $f_n\to f^*$ as $n\to\infty$. By employing (4), we have,

$$\zeta(f^*,\tau f^*)\leqslant\alpha(f^*,\tau f_n)\zeta(f^*,\tau f_n)+\beta(\tau f_n,\tau f^*)\zeta(\tau f_n,\tau f^*)$$
$$\leqslant\alpha(f^*,\tau f_n)\zeta(f^*,\tau f_n)+\beta(\tau f_n,\tau f^*)[a\zeta(f_n,\tau f_n)+b\zeta(f^*,\tau f^*)],$$

which implies

$$(1-b\beta(\tau f_n,\tau f^*))\zeta(f^*,\tau f^*)\leqslant\alpha(f^*,\tau f_n)\zeta(f^*,\tau f_n)+a\beta(\tau f_n,\tau f^*)\zeta(f_n,\tau f_n).$$

From the fact that $0<b<1$, $\beta:M\times M\to[1,\frac{1}{b})$ and both the limits $\lim\limits_{n\to\infty}\alpha(f,f_n)$ and $\lim\limits_{n\to\infty}\beta(f,f_n)$ exist and are finite, we get $\zeta(f^*,\tau f^*)=0$, so $f^*=\tau f^*$. Similar to Theorem 3, the conclusion is true.

Corollary 8 Let $\tau:M\to M$ be an asymptotically regular mapping on the τ-orbitally complete DCMLS. The mapping τ satisfies

$$\zeta(\tau f,\tau g)\leqslant k[\zeta(f,\tau f)+\zeta(g,\tau g)], \tag{5}$$

for all $f,g\in M$, where $k\in(0,1)$. If for any $f,f_0\in M$, $f_n=\tau^n f_0$, and the noncomparable functions $\alpha:M\times M\to[1,+\infty),\beta:M\times M\to[1,\frac{1}{k})$, both the limits $\lim\limits_{n\to\infty}\alpha(f,f_n)$ and $\lim\limits_{n\to\infty}\beta(f,f_n)$ exist and are finite, then τ has a unique fixed point $f^*\in M$ and for each $f\in M$ the iterated sequence $\tau^n f$ converges to f^*, i.e., τ is a Picard operator.

Remark 2 Theorem 3 is an improvement of Corollary 2.10 in Ref. [19]. Similarly, Theorem 4 is an extension of Theorem 3.1, 3.8 in Ref. [15], since our spaces are τ-orbitally complete DCMLS instead of complete metric spaces. In addition, we establish the contractive mappings with generalized Lipschitz constants, where the constants $a,b\in[0,+\infty)$, instead of limiting in $[0,\frac{1}{2})$. Moreover, the continuity of the mapping is weakened by weak orbital continuity in the theorems which is necessary in Theorem 3.3 of Ref. [22], Corollary 2.10 of Ref. [19].

Example 4 Let $M=[0,2)$ and

$$\zeta(f,g)=\begin{cases}1, & f=g=0,\\ |f-g|^2, & \text{otherwise},\end{cases}$$

for all $f,g\in M$. Define the noncomparable functions α,β: $M\times M\to\mathbb{R}^+$ by

$$\alpha(f,g)=\begin{cases}\left|\dfrac{2(f+g)}{f-g}\right|, & f\neq g,\\ 2, & \text{otherwise},\end{cases}$$

and
$$\beta(f,g)=\begin{cases}\dfrac{1}{f^2}+1, & f\neq 0, g=0,\\ \dfrac{1}{g^2}+1, & f=0, g\neq 0,\\ 2, & \text{otherwise}.\end{cases}$$

At first, we check that (M,ζ) is a DCMLS. Conditions 1) and 2) of Definition 1 hold by a simple verification. Now, we infer that it meets the condition 3) in four cases.

1) If $f=g=0$, then

$$\zeta(f,g)=1\leqslant\begin{cases}2\times1+2\times1, & h=0,\\ 2\cdot h^2+\left(\dfrac{1}{h^2}+1\right)\cdot h^2, & h\neq0.\end{cases}$$

2) If $f=0, g\neq0$, then

$$\zeta(f,g)=g^2\leqslant\begin{cases}2\times1+\left(\dfrac{1}{g^2}+1\right)\cdot g^2, & h=0,\\ 2\cdot h^2+2\left|h-g\right|^2, & h\neq0.\end{cases}$$

3) If $f\neq0, g=0$, then

$$\zeta(f,g)=f^2\leqslant\begin{cases}2\cdot f^2+2\times1, & h=0,\\ 2\times0+\left(\dfrac{1}{h^2}+1\right)\cdot h^2, & h\neq0, f=h,\\ \left|\dfrac{2(f+h)}{f-h}\right|\cdot\left|f-h\right|^2+\left(\dfrac{1}{h^2}+1\right)\cdot h^2, & h\neq0, f\neq h.\end{cases}$$

For the third inequality, if $f>h$, we have

$$\left|\frac{2(f+h)}{f-h}\right|\cdot\left|f-h\right|^2+\left(\frac{1}{h^2}+1\right)\cdot h^2-f^2$$
$$=2|f^2-h^2|+h^2+1-f^2=f^2-h^2+1>0.$$

If $f<h$, then

$$f^2\leqslant h^2+1\leqslant\left|\frac{2(f+h)}{f-h}\right|\cdot\left|f-h\right|^2+\left(\frac{1}{h^2}+1\right)\cdot h^2$$

holds obviously.

4) If $f\neq0, g\neq0$, then

$$\zeta(f,g)=|f-g|^2\leqslant\begin{cases}2f^2+\left(\dfrac{1}{g^2}+1\right)\cdot g^2, & h=0,\\ 2\times0+2\cdot\left|h-g\right|^2, & h\neq0, f=h,\\ \left|\dfrac{2(f+h)}{f-h}\right|\cdot\left|f-h\right|^2+2\cdot\left|h-g\right|^2, & h\neq0, f\neq h.\end{cases}$$

Since $\left|\dfrac{2(f+h)}{f-h}\right|\geqslant2$, the last inequality holds. Thus, (M,ζ) is a DCMLS but not a DCMTS while $f=g=0$, $\zeta(f,g)\neq0$, that is $f=g\not\Leftrightarrow\zeta(f,g)=0$. It is straightforward to show that (M,ζ) is not a CMTS or an EbMS. Moreover, (M,ζ) is not complete.

Next, choose $a,b\geqslant\dfrac{1}{3}$, define the mapping $\tau:M\to M$ by:

$$\tau f=\begin{cases}1, & f\in[0,\dfrac{4}{3}),\\ \dfrac{4+f}{7}, & f\in[\dfrac{4}{3},2).\end{cases}$$

Then, τ is asymptotically regular and weakly orbitally continuous but not orbitally continuous or k-continuous. Indeed, for any $f\in[\frac{4}{3},2)$,

$$\tau^n(f)=\frac{2}{3}-\frac{2}{3}\left(\frac{1}{7}\right)^n+\left(\frac{1}{7}\right)^n f\to\frac{2}{3}\neq\tau\frac{2}{3}=1,$$

which shows τ is not orbitally continuous. We can deduce that τ is not k-continuous in the same manner. At last, let us show that the inequality (2) is satisfied in the following three cases.

1) For all $f,g\in[0,\frac{4}{3})$, $\tau f=\tau g=1$, $\zeta(\tau f,\tau g)=0$, then it is true.

2) For all $f,g\in[\frac{4}{3},2)$, $\tau f=\dfrac{4+f}{7}$, $\tau g=\dfrac{4+g}{7}$,

$$\zeta(\tau f,\tau g)=\left|\frac{f-g}{7}\right|^2\leqslant\frac{1}{3}\left|\frac{6f-4}{7}\right|^2+\frac{1}{3}\left|\frac{6g-4}{7}\right|^2$$
$$\leqslant a\left|f-\frac{4+f}{7}\right|^2+b\left|g-\frac{4+g}{7}\right|^2$$
$$=a\zeta(f,\tau f)+b\zeta(g,\tau g).$$

For the second inequality, we only need to prove

$$\frac{1}{3}(6f-4)^2+\frac{1}{3}(6g-4)^2>(f-g)^2\geqslant0.$$

In fact, for $f,g\in[\frac{4}{3},2)$,

$$(f-g)^2\leqslant\left|\frac{4}{3}-2\right|^2=\frac{4}{9}<\frac{16}{3}<\frac{1}{3}(6f-4)^2+\frac{1}{3}(6g-4)^2.$$

Hence, (2) holds.

3) For all $f\in[0,\frac{4}{3})$, $g\in[\frac{4}{3},2)$, $\tau f=1$, $\tau g=\dfrac{4+g}{7}$, we have

$$\zeta(\tau f,\tau g)=\left|1-\frac{4+g}{7}\right|^2=\left|\frac{3-g}{7}\right|^2$$
$$\leqslant\frac{1}{3}\left|f-1\right|^2+\frac{1}{3}\left|\frac{6g-4}{7}\right|^2$$
$$\leqslant a\left|f-1\right|^2+b\left|g-\frac{4+g}{7}\right|^2$$
$$=a\zeta(f,\tau f)+b\zeta(g,\tau g).$$

Similarly, for the third inequality, we have

$$\frac{1}{3}(6g-4)^2-(g-3)^2>0.$$

In fact, $\frac{1}{3}(6g-4)^2\geqslant\frac{16}{3}>\frac{25}{9}\geqslant(g-3)^2$, for $g\in[\frac{4}{3},2)$. Thus, (2) holds.

By similar analysis, (2) holds for all $f\in[\frac{4}{3},2)$, $g\in[0,\frac{4}{3})$. Therefore, τ has a unique fixed point by Theorem 3. Since $a,b\in[\frac{1}{3},+\infty)$, (M,ζ) is incomplete and τ is not continuous or orbitally continuous, we can conclude that those known results in the literature that need these conditions can not cope with this class of examples, indicating that the main results in our paper are genuine generalizations of the known results.

4 Completeness and Fixed Point

In this section, we show the relationship between the completeness of DCMLS and the existence of fixed points. In other words, the necessary and sufficient condition for the existence of a fixed point in DCMLS is that the spaces must be complete.

Theorem 5 Let (M,ζ) be a DCMLS. For any $f, f_0 \in M$, define $f_n = \tau^n f_0, n \geqslant 1$. The limits $\lim\limits_{n,m\to\infty} \alpha(f_m, f_n)$ and $\lim\limits_{n\to\infty} \beta(f, f_n)$ exist and are finite. If every mapping τ: $M \to M$ satisfies

$$\zeta(\tau f, \tau g) < a\zeta(f, \tau f) + b\zeta(g, \tau g), \tag{6}$$

for all $f, g \in M$ with $f \neq g$ and $a+b=1, a, b \in (0,1)$, it has a unique fixed point, then (M,ζ) must be a complete DCMLS.

Proof Proving by contradiction. Suppose that (M,ζ) is not complete, then we have a ζ-Cauchy sequence $\{f_n\}$ in M, which is not convergent. If $\{f_n\}$ has a convergent subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that $f_{n_i} \to f^* \in M$, then for all $n \geqslant n_i$,

$$\zeta(f_n, f^*) \leqslant \alpha(f_n, f_{n_i})\zeta(f_n, f_{n_i}) + \beta(f_{n_i}, f^*)\zeta(f_{n_i}, f^*),$$

which convergent to 0 for $i \to \infty$. This means we can assume that all elements of $\{f_n\}$ are distinct. Let $S=\{f_n$: $n \in \mathbb{N}\}$. Since the sequence $\{f_n\}$ does not converge in M, we obtain $\zeta(f, S) > 0$ for all $f \in M-S$. For any $f \in M$, if $f \in M-S$, then there is an integer $n_f \in \mathbb{N}$ such that $\zeta(f_m, f_{n_f}) < a\zeta(f,S) \leqslant a\zeta(f, f_n)$, for all $m \geqslant n_f$ and arbitrary $n \in \mathbb{N}$, that is

$$\zeta(f_m, f_{n_f}) < a\zeta(f, f_n), \forall m \geqslant n_f \text{ and } \forall n \in \mathbb{N}. \tag{7}$$

If $f' \in S$, then $f' = f_{n_0}$ for some $n_0 \in \mathbb{N}$. Since $\{f_n\}$ is a ζ-Cauchy sequence, there exists some $n_0' \in \mathbb{N}$ such that

$$\zeta(f_m, f_{n_0'}) < b\zeta(f_{n_0'}, f_{n_0}), \forall m \geqslant n_0' > n_0. \tag{8}$$

Define τ:$M \to M$ by

$$\tau f = \begin{cases} f_{n_f}, & \text{if } f \in M-S, \\ f_{n_0'}, & \text{if } f \in S \text{ and } f = f_{n_0}. \end{cases}$$

Now, it is enough to check (6). There are three cases.

Case 6 If $f, g \in M-S$, then $\tau f = f_{n_f}$ and $\tau g = f_{n_g}$. Without loss of generality, we assume that $n_g \geqslant n_f$. By (7), we get

$$\zeta(\tau f, \tau g) = \zeta(f_{n_f}, f_{n_g}) = \zeta(f_{n_g}, f_{n_f}) < a\zeta(f, f_{n_f}) = a\zeta(f, \tau f),$$

that is, $\zeta(\tau f, \tau g) < a\zeta(f, \tau f) + b\zeta(g, \tau g)$.

Case 7 If $f, g \in M-S$, then $f = f_{m_0}$ and $g = f_{n_0}$ for some $m_0, n_0 \in \mathbb{N}$, and $\tau f = f_{m_0'}, \tau g = f_{n_0'}$. Without loss of generality, we assume that $m_0' \geqslant n_0'$. By (8), we have

$$\zeta(\tau f, \tau g) = \zeta(f_{m_0'}, f_{n_0'}) < b\zeta(f_{n_0'}, f_{n_0}) = b\zeta(\tau g, g),$$

which implies that $\zeta(\tau f, \tau g) < a\zeta(f, \tau f) + b\zeta(g, \tau g)$.

Case 8 If $f \in M-S$, $g \in S$, then $g = f_{n_0}$ for some $n_0 \in \mathbb{N}$, and $\tau f = f_{n_f}$, $\tau g = f_{n_0'}$. If $n_0' \geqslant n_f$, by inequality (7), we get

$$\zeta(\tau f, \tau g) = \zeta(\tau g, \tau f) = \zeta(f_{n_0'}, f_{n_f}) < a\zeta(f, f_{n_f}) = a\zeta(f, \tau f),$$

which implies that $\zeta(\tau f, \tau g) < a\zeta(f, \tau f) + b\zeta(\tau g, g)$. If $n_f > n_0'$, inequality (8) yields

$$\zeta(\tau f, \tau g) = \zeta(f_{n_f}, f_{n_0'}) < b\zeta(f_{n_0'}, f_{n_0}) = b\zeta(g, \tau g),$$

which also gives that $\zeta(\tau f, \tau g) < a\zeta(f, \tau f) + b\zeta(g, \tau g)$. Combining the above three situations, we always have $\zeta(\tau f, \tau g) < a\zeta(f, \tau f) + b\zeta(g, \tau g)$ for all $f, g \in M$ with $f \neq g$, which means τ satisfies (6) but has no fixed point in M. It is a contradiction. Therefore, the initial assumption is not right and the space (M,ζ) must be complete.

From the proof process of Theorem 5, we immediately conclude that if the spaces are replaced by b-metric spaces or metric spaces, the condition "the limits $\lim\limits_{n,m\to\infty} \alpha(f_m, f_n)$ and $\lim\limits_{n\to\infty} \beta(f, f_n)$ exist and are finite" can be omitted.

Corollary 9 Let (M,ζ) be a b-metric space. If every mapping τ:$M \to M$ satisfies

$$\zeta(\tau f, \tau g) < a\zeta(f, \tau f) + b\zeta(g, \tau g),$$

for all $f, g \in M$ with $f \neq g$ and $a+b=1, a, b \in (0,1)$, it has a unique fixed point, then (M,ζ) must be a complete b-metric space.

Remark 3 Theorem 5 and Corollary 9 generalize the classic theorems in Ref. [18], since $a = b = \frac{1}{2}$ is extended to $a+b=1, a, b \in (0,1)$ and the metric spaces are extended to b-metric spaces and DCMLS.

5 Conclusion

In this paper, we investigate fixed point theorems for extended Kannan-type contractions in DCMLS, providing significant generalizations of existing results. We introduce some concepts such as bounded compactness, τ-orbital compactness, weak orbital continuity and asymptotic regularity in DCMLS, which relax the traditional requirements of completeness and continuity. The key contributions include establishing the existence and uniqueness of fixed points under weaker conditions, demonstrating that completeness is necessary for the existence of fixed points in DCMLS and illustrating the applicability of these results through concrete examples. The work extends classical theorems by generalizing the contraction constants and broadening the scope to various generalized metric spaces.

In the future, we may consider further investigating fixed points for Hardy-Rogers-type and Ciric-type contractions in DCMLS. Additionally, applications to nonlinear integral equations and differential equations could be studied. Another promising direction is to extend DCMLS to more abstract settings, such as double controlled cone metric-like spaces over Banach algebras.

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广义度量空间中各类弱条件下压缩映射的不动点定理

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摘要：本文通过提出一些新的定义来研究双控制度量类空间中的几个不动点定理，这些定义是作为现有概念的推广。同时，通过一些典型例子说明了主要结果的应用。具体而言，这些实例适用于双控制度量类空间，而非通常的 *b*-度量空间。我们在这些新概念下获得了不动点定理，不要求空间的完备性或映射的连续性。此外，证明了在双控制度量类空间中，如果推广的 Kannan 型压缩映射在空间中有不动点，则该空间的完备性是必要的。本文的结果有助于更深入地理解 Kannan 型压缩映射及其在双控制度量类空间不动点理论中的应用。

关键词：双控制度量类空间；完备性；渐近正则性；不动点

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