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# Complete Convergence for the Maximum Partial Sums of *m*-Widely Orthant Dependent Random Variables Sequences

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**Abstract:** In this paper, the author obtains complete convergence for the maximum partial sums of *m*-widely orthant dependent (*m*-WOD) random variables sequences under some general conditions. The results extend the complete convergence for *m*-WOD random variables to a much more general type complete convergence. As the sequences of *m*-WOD random variables represent a very broad class of dependent sequences, the results improve and generalize the corresponding ones in the literature.

**Key words:** *m*-WOD random variables; complete convergence; Spitzer's law of large numbers

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## 0 Introduction

In many probabilistic and statistical models, random variables are dependent. Consequently, scholars have introduced many types of dependent random variables, such as negatively associated (NA) random variables, negatively orthant dependent (NOD) random variables, and extend negatively dependent (END) random variables and so on. Among these, widely orthant dependent (WOD) random variables represent one of the most general forms of dependence. They were first introduced by Wang *et al*[1], defined as follows:

**Definition 1** The random variables $\{X_n, n\geqslant 1\}$ are called to be widely upper orthant dependent (WUOD) random variables, if there exist finite sequences of real numbers $\{g_U(n), n\geqslant 1\}$ such that for each $n\geqslant 1, x_1, x_2, \cdots, x_n\in\mathbb{R}$,

$$P(X_1>x_1, X_2>x_2, \cdots, X_n>x_n)\leqslant g_U(n)\prod_{i=1}^{n}P(X_i>x_i).$$

The random variables $\{X_n, n\geqslant 1\}$ are called to be widely lower orthant dependent (WLOD) random variables, if there exist finite sequences of real numbers $\{g_L(n), n\geqslant 1\}$ such that for each $n\geqslant 1$, $x_1, x_2, \cdots, x_n\in\mathbb{R}$,

$$P(X_1\leqslant x_1, X_2\leqslant x_2, \cdots, X_n\leqslant x_n)\leqslant g_L(n)\prod_{i=1}^{n}P(X_i>x_i).$$

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The random variables $\{X_n, n \geqslant 1\}$ are called to be widely orthant dependent (WOD) random variables, if the random variables $\{X_n, n \geqslant 1\}$ are both WUOD and WLOD. Let $g(n) = \max\{g_U(n), g_L(n)\}$ be called dominated coefficients.

When $g_U(n) = g_L(n) = 1$, $\{X_n, n \geqslant 1\}$ are NOD random variables[2]. When $g_U(n) = g_L(n) = M \geqslant 1$, $\{X_n, n \geqslant 1\}$ are END random variables[3]. Therefore, WOD random variables represent a broad structure of dependent random variables.

Since the concept of WOD random variables was introduced, many scholars have devoted efforts to studying their limit properties and applications, achieving significant results. For example, Wang *et al*[4] obtained the precise large deviations; Qiu *et al*[5] established the complete convergence and moment complete convergence of the weighted sums; Liu *et al*[6] derived the moment complete convergence; Wang *et al*[7] and Chen *et al*[8] studied the asymptotic of ruin probabilities in renewal risk models based on WOD sequences; Shen[9] proved the Bernstein-type probability inequality; Wang *et al*[10] investigated complete convergence and its applications in nonparametric regression models; Ding *et al*[11] provided results on the complete convergence of weighted sums; Song *et al*[12-14] analyzed the convergence of moving average processes generated by WOD random variables, and so on.

Inspired by $m$-END and WOD dependence structures, Fang *et al*[15] introduced the concept of $m$-WOD random variables, defined as follows:

**Definition 2** For fix integer $m \geqslant 1$, the random variables $\{X_n, n \geqslant 1\}$ are called to be $m$-WOD if for any $n \geqslant 2$, $i_1, i_2, \cdots, i_n \in \mathbb{N}^+$, such that $|i_k - i_j| \geqslant m$ for all $1 \leqslant k \neq j \leqslant n$, the $X_{i_1}, X_{i_2}, \cdots, X_{i_n}$ are also WOD random variables.

From the definition, we see that $m$-WOD random variables represent a broader class of dependence than WOD random variables. Therefore, investigating the complete convergence of $m$-WOD random variables is very interesting.

It is well-known that for sequencse of independent and identically distributed random variables $\{X_n, X, n \geqslant 1\}$, Spitzer[16] proved that $EX = 0$ is equivalent to

$$\sum_{n=1}^{\infty} n^{-1} P\left(|\sum_{j=1}^{n} X_j| > \varepsilon n\right) < \infty, \ \forall \varepsilon > 0. \tag{1}$$

and (1) is equivalent to

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leqslant k \leqslant n} |\sum_{j=1}^{k} X_j| > \varepsilon n\right) < \infty, \ \forall \varepsilon > 0. \tag{2}$$

However, the converse does not hold for the dependent case, as shown in Ref. [17]. Therefore, it is more interesting to investigate (2) than (1), and we note that (2) implies Kolmogorov's strong law of large numbers

$$\frac{1}{n} \sum_{j=1}^{n} X_j \to 0, \quad \text{a.s.}$$

Recently, Chen *et al*[18-19] obtained Spitzer's law for maximum partial sums of WOD random variables. Inspired by the above study, this paper aims to generalize Chen's results[18-19] to cases of complete convergence for $m$-WOD random variables.

**Definition 3** The random variables $\{X_n, n \geqslant 1\}$ are called to be stochastically dominated by a random variable $X$, if for any $x > 0$,

$$P(|X_n| > x) \leqslant CP(|X| > x), \ n \geqslant 1,$$

where the constant $C > 0$.

In this paper, $I(A)$ denotes the indicator function of an event $A$, the symbol $C$ represents a positive constant, which can take different values in different places, even in the same formula. Let $\log n = \ln\max\{x, \mathrm{e}\}$, $X^+ = XI(X > 0)$, $g(n) = \max\{g_U(n), g_L(n)\}$.

## 1 Some Lemmas and Main Results

**Lemma 1**[15] The sequences $\{X_n, n \geqslant 1\}$ are $m$-WOD random variables, if the functions $\{f_n, n \geqslant 1\}$ are non-decreasing (non-increasing), then $\{f_n(X_n), n \geqslant 1\}$ are also $m$-WOD random variables sequences with same dominating coefficients.

**Lemma 2**[15] The sequences $\{X_n,\ n\geqslant 1\}$ are $m$-WOD random variables with dominating coefficients $g(n)$. For every $j\geqslant 1$, the $EX_j=0$ and $E|X_j|^p<\infty$. Then, there exist positive constants $C_1=C_1(p,m),\ C_2=C_2(p,m)$, depending only on $p$ and $m$, such that

$$E\left(\sum_{j=1}^{n}|X_j|^p\right)\leqslant\left[C_1(p,m)+C_2(p,m)g(n)\right]\sum_{j=1}^{n}E|X_j|^p,\ 1<p\leqslant 2;$$

$$E\left(\sum_{j=1}^{n}|X_j|^p\right)\leqslant C_1(p,m)\sum_{j=1}^{n}E|X_j|^p+C_2(p,m)g(n)\left(\sum_{j=1}^{n}EX_j^2\right)^{p/2},\ p>2.$$

**Lemma 3**[15] The sequences $\{X_n,\ n\geqslant 1\}$ are $m$-WOD random variables with dominating coefficients $g(n)$. For every $j\geqslant 1$, the $EX_j=0$ and $E|X_j|^p<\infty$. Then, there exist positive constants $C_1=C_1(p,m),\ C_2=C_2(p,m)$, depending only on $p$ and $m$, such that

$$E\left(\max_{1\leqslant k\leqslant n}\sum_{j=1}^{k}|X_j|^p\right)\leqslant\left[C_1(p,m)+C_2(p,m)g(n)\right](\log n)^p\sum_{j=1}^{n}E|X_j|^p,\ 1<p\leqslant 2;$$

$$E\left(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}X_j|^p\right)\leqslant C_1(p,m)(\log n)^p\sum_{j=1}^{n}E|X_j|^p+C_2(p,m)g(n)(\log n)^p\left(\sum_{j=1}^{n}EX_j^2\right)^{p/2},\ p>2.$$

**Lemma 4**[20] Constant $a>0, b>0$, let $\{X_n, n\geqslant 1\}$ be stochastically dominated by $X$, then there exist positive constants $C_1, C_2$ such that the following inequalities are established:

$$E|X_n|^a I(|X_n|\leqslant b)\leqslant C_1\left[E|X|^a I(|X|\leqslant b)+b^a I(|X|>b)\right],$$

$$E|X_n|^a I(|X_n|>b)\leqslant C_2 E|X|^a I(|X|>b).$$

Now, we present the main results, the proofs for them will be postponed in next section.

**Theorem 1** Let $\{X_n,\ n\geqslant 1\}$ be sequences of $m$-WOD random variables stochastically dominated by a random variable $X$ with dominating coefficients $g(n)$. Let $0<1/p\leqslant\alpha<1,\ E|X|^p<\infty$. Assume that one of the following conditions holds:

$A_1$: Let $g(x)$ be a nondecreasing positive function on $[0,+\infty)$, such that $g(x)/x^\tau\downarrow 0$ for some $\tau>0$.

$A_2$: Let $h(x)$ be a nondecreasing positive function on $[0,+\infty)$, such that $h(x)/x\downarrow 0$ and $\sum_{n=1}^{\infty}\frac{g(n)}{nh^\gamma(n)}<\infty$ for some $\gamma>0$.

Then, for $\theta>\max\{1/2,\ \alpha\}$,

$$\sum_{n=1}^{\infty}n^{\alpha p-2}P\left(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}(X_j-EX_j)|>\varepsilon n^\theta\right)<\infty,\quad\forall\varepsilon>0. \tag{3}$$

If $\alpha=1$, we have the following result.

**Theorem 2** Let $\{X_n,\ n\geqslant 1\}$ be sequences of $m$-WOD random variables stochastically dominated by a random variable $X$ with dominating coefficients $g(n)$, $E|X|^{p+\delta}<\infty,\ p>1,\ 0<\delta<p(p-1)$. Assume $A_1$ or $A_2$ holds. Then, for $\theta>\max\{1/2,\ p/(p+\delta)\}$,

$$\sum_{n=1}^{\infty}n^{p-2}P\left(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}(X_j-EX_j)|>\varepsilon n^\theta\right)<\infty,\quad\forall\varepsilon>0. \tag{4}$$

**Theorem 3** Let $\{X_n, n\geqslant 1\}$ be sequences of $m$-WOD random variables stochastically dominated by a random variable $X$ with dominating coefficients $g(n)$, $E|X|^p<\infty,\ p>1$. Assume $A_1$ or $A_2$ holds. Then

$$\sum_{n=1}^{\infty}n^{p-2}P(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}(X_j-EX_j)|>\varepsilon n)<\infty,\quad\forall\varepsilon>0. \tag{5}$$

If $\alpha p=1$, we have the following result.

**Theorem 4** Let $\{X_n,\ n\geqslant 1\}$ be sequences of $m$-WOD random variables stochastically dominated by a random variable $X$ with dominating coefficients $g(n)$, $E|X|h^\delta(|X|)<\infty,\ 0<\delta\leqslant 1$. Assume $A_2$ holds and

$$\limsup_{x\to\infty}\frac{h(x)}{h(x/h(x))}<\infty, \tag{6}$$

then

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}(X_j-EX_j)|>\varepsilon n\right)<\infty,\quad \forall\varepsilon>0. \tag{7}$$

**Remark 1** The $m$-WOD random variables encompass WOD, $m$-NA, $m$-NOD, $m$-END, among others. Thus, the results in this paper extend and improve upon existing results.

**Remark 2** Since stochastic domination is a weaker condition than identical distribution, the results in this paper also hold under the condition of identical distribution.

**Remark 3** Taking $\delta=1$ in Theorem 4, we obtain the result of Theorem 1.1 in Ref. [17]. Taking $\delta=1/2$, $h(x)=g(x)$ in Theorem 4, we obtain the result of Theorem 1.2 in Chen *et al*'s [18]. Therefore, our results extend and improve the results in Refs. [17-18].

## 2 Proof of Theorems

**Proof of Theorem 1** Noting $X_i=X_i^+-X_i^-$, hence to prove (3), it only to prove that, for any $\varepsilon>0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}(X_j^+-EX_j^+)|>\varepsilon n^{\theta}/2\right)<\infty,\quad \forall\varepsilon>0,$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}(X_j^--EX_j^-)|>\varepsilon n^{\theta}/2\right)<\infty,\quad \forall\varepsilon>0.$$

By Lemma 1, $\{X_n^+,\ n\geqslant 1\}$ and $\{X_n^-,\ n\geqslant 1\}$ are also $m$-WOD random variables sequences with dominating coefficients $g(n)$, $n\geqslant 1$. Therefore, without loss of generality, we assume $X_n\geqslant 0$, $n\geqslant 1$.

For a fixed $n\geqslant 1$, for $1\leqslant j\leqslant n$, denote

$$Y_{nj}=X_jI\left(X_j\leqslant n^{\alpha}\right)+n^{\alpha}I\left(X_j>n^{\alpha}\right),\ Z_{nj}=X_j-Y_{nj}=\left(X_j-n^{\alpha}\right)I\left(X_j>n^{\alpha}\right).$$

Then

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}(X_j-EX_j)|>\varepsilon n^{\theta}\right) &\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}(Y_{nj}-EY_{nj})|>\varepsilon n^{\theta}/2\right)+\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\bigcup_{j=1}^{n}\{X_j>n^{\alpha}\}\right)\\ &\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}(Y_{nj}-EY_{nj})|>\varepsilon n^{\theta}/2\right)+\sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{j=1}^{n} P\left(X_j>n^{\alpha}\right)=:I_1+I_2.\end{aligned}$$

Thus, to prove (3), we only need to show $I_1<\infty$ and $I_2<\infty$.

Combining with Lemma 4 and condition $\alpha p-1>-1$, we obtain

$$\begin{aligned}I_2 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1} P\left(X>n^{\alpha}\right)=C\sum_{n=1}^{\infty} n^{\alpha p-1}\sum_{k=n}^{\infty} P\left(k^{\alpha}<X\leqslant(k+1)^{\alpha}\right)\\ &=C\sum_{k=1}^{\infty} P\left(k^{\alpha}<X\leqslant(k+1)^{\alpha}\right)\sum_{n=1}^{k} n^{\alpha p-1}\leqslant C\sum_{k=1}^{\infty} k^{\alpha p} P\left(k^{\alpha}<X\leqslant(k+1)^{\alpha}\right)\leqslant CE|X|^p<\infty.\end{aligned} \tag{8}$$

By Lemma 1, $\left\{Y_{nj}-EY_{nj}\right\}$ are also $m$-WOD random variables with the same dominating coefficients.

For $I_1$, by Markov's inequality and Lemma 3 , we have that for any $v>\max\{2,p\}$,

$$I_1\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\theta v} E\left\{\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}(Y_{nj}-EY_{nj})|^v\right\}\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\theta v}\left(\log n\right)^v\left\{\sum_{j=1}^{n}E|Y_{nj}|^v+g\left(n\right)\left(\sum_{j=1}^{n}E|Y_{nj}|^2\right)^{v/2}\right\}=:I_{11}+I_{12}. \tag{9}$$

For $I_{11}$, combining with Lemma 4 and the condition $\theta>\{1/2,\alpha\}$, we obtain

$$\begin{aligned}I_{11} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\theta v}\left(\log n\right)^v\sum_{j=1}^{n}\left[EX_j^vI\left(X_j\leqslant n^{\alpha}\right)+n^{v\alpha}P\left(X_j>n^{\alpha}\right)\right]\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1-\theta v}\left(\log n\right)^v\left[EX^vI\left(X\leqslant n^{\alpha}\right)+n^{v\alpha}P\left(X>n^{\alpha}\right)\right]\\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1-\theta v}\left(\log n\right)^v\left[EX^pn^{\alpha(v-p)}I\left(X\leqslant n^{\alpha}\right)+EX^pn^{\alpha(v-p)}P\left(X>n^{\alpha}\right)\right]\leqslant C\sum_{n=1}^{\infty} n^{-1-(\theta-\alpha)v}\left(\log n\right)^v<\infty.\end{aligned} \tag{10}$$

For $I_{12}$, we have

$$I_{12}\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2-\theta v}\left(\log n\right)^{v} g(n)\left\{\sum_{j=1}^{n}\left[EX_j^{2} I\left(X_j\leqslant n^{\alpha}\right)+n^{2\alpha}P\left(X_j>n^{\alpha}\right)\right]\right\}^{v/2}$$
$$\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\theta v+v/2}\left(\log n\right)^{v} g(n)\left[EX^{2} I\left(X\leqslant n^{\alpha}\right)+n^{2\alpha}P\left(X>n^{\alpha}\right)\right]^{v/2}. \qquad (11)$$

To prove $I_{12}<\infty$, we consider two cases:

**Case 1:** When $p\geqslant 2$, the condition $E|X|^p<\infty$ implies $E|X|^2<\infty$ , taking $v>\max\left\{p, \dfrac{2\left(\alpha p-1+\tau\right)}{2\theta-1}, \dfrac{2\left(\alpha p+\gamma\right)}{2\theta-1}\right\}$, then

$$I_{12}\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\theta v+v/2}\left(\log n\right)^{v} g(n)\left[EX^{2}\right]^{v/2}\leqslant \begin{cases} C\sum_{n=1}^{\infty} n^{\alpha p-2-\theta v+v/2+\tau}\left(\log n\right)^{v}, \text{ if } A_1 \text{ holds} \\ C\sum_{n=1}^{\infty} n^{\alpha p-2-\theta v+v/2+1+\gamma}\left(\log n\right)^{v}\dfrac{g(n)}{nh^{\gamma}(n)}, \text{ if } A_2 \text{ holds}\end{cases}<\infty. \qquad (12)$$

**Case 2:** When $1<p<2$, under $E|X|^p<\infty$ , taking $v>\max\left\{2, \dfrac{2\left(\alpha p-1+\tau\right)}{2\left(\theta-\alpha\right)+\left(\alpha p-1\right)}, \dfrac{2\left(\alpha p+\gamma\right)}{2\left(\theta-\alpha\right)+\left(\alpha p-1\right)}\right\}$, we obtain

$$I_{12}\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\theta v+v/2}\left(\log n\right)^{v} g(n)\left[EX^{p} n^{\alpha(2-p)} I\left(X\leqslant n^{\alpha}\right)+EX^{p} n^{\alpha(2-p)} P\left(X>n^{\alpha}\right)\right]^{v/2}$$
$$\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\theta v+v/2}\left(\log n\right)^{v} g(n) n^{\alpha(2-p)v/2}\left(EX^{p}\right)^{v/2}\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\theta v+v/2+\alpha v-\alpha p v/2}\left(\log n\right)^{v} g\left(n\right)$$
$$\leqslant \begin{cases} C\sum_{n=1}^{\infty} n^{\alpha p-2-\theta v+v/2+\alpha v-\alpha p v/2+\tau}\left(\log n\right)^{v}, \text{ if } A_1 \text{ holds} \\ C\sum_{n=1}^{\infty} n^{\alpha p-2-\theta v+v/2+\alpha v-\alpha p v/2+1+\gamma}\left(\log n\right)^{v}\dfrac{g(n)}{nh^{\gamma}(n)}, \text{ if } A_2 \text{ holds}\end{cases}<\infty. \qquad (13)$$

From (8)-(13), the proof of Theorem 1 is completed.

**Proof of Theorem 2** For fixed $n\geqslant 1$, $1\leqslant j\leqslant n$, denote

$$Y_{nj}=X_j I\left(X_j\leqslant n^{p/(p+\delta)}\right)+n^{p/(p+\delta)} I\left(X_j>n^{p/(p+\delta)}\right),$$
$$Z_{nj}=X_j-Y_{nj}=\left(X_j-n^{p/(p+\delta)}\right) I\left(X_j>n^{p/(p+\delta)}\right).$$

The method and the proof for Theorem 2 are the same as Theorem 1, so they are omitted.

**Proof of Theorem 3** The condition $E|X|^p<\infty$, $p>1$, implies that $E|X|<\infty$ . Consequently, there exists a positive integer $N$ such that $EXI(X>N)<\varepsilon/8$.

For $j\geqslant 1$, let

$$Y_j=X_j I\left(X_j\leqslant N\right)+NI\left(X_j>N\right),\ Z_j=X_j-Y_j=\left(X_j-N\right) I\left(X_j>N\right).$$

Then

$$\sum_{n=1}^{\infty} n^{p-2} P\left(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}(X_j-EX_j)|>\varepsilon n\right)\leqslant \sum_{n=1}^{\infty} n^{p-2} P\left(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}(Y_j-EY_j)|>\varepsilon n/2\right)+\sum_{n=1}^{\infty} n^{p-2} P\left(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}(Z_j-EZ_j)|>\varepsilon n/2\right)$$
$$=:I_3+I_4. \qquad (14)$$

Thus, to prove (14), we only need to show that $I_3<\infty$ and $I_4<\infty$.

For $I_3$, noting $\left\{(Y_j-EY_j), j\geqslant 1\right\}$ are sequences of $m$-WOD with dominating coefficient $g(n)$ for $n\geqslant 1$ by Lemma 1. By Markov's inequality, Lemma 3 and Lemma 4, taking $v>\max\left\{p, 2, 2(p-1+\tau), 2(p+\gamma)\right\}$, we have

$$I_3\leqslant C\sum_{n=1}^{\infty} n^{p-2-v} E\left\{\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}(Y_j-EY_j)|^{v}\right\}\leqslant C\sum_{n=1}^{\infty} n^{p-2-v}\left(\log n\right)^{v}\left\{\sum_{j=1}^{n}E|Y_j|^{v}+g\left(n\right)\left(\sum_{j=1}^{n}|EY_j|^{2}\right)^{v/2}\right\}$$
$$\leqslant C\sum_{n=1}^{\infty} n^{p-2-v}\left(\log n\right)^{v}\sum_{j=1}^{n}\left[EX_j^{v} I\left(X_j\leqslant N\right)+N^{v}P\left(X_j>N\right)\right]$$

$$+C\sum_{n=1}^{\infty}n^{p-2-v}\left(\log n\right)^{v}g\left(n\right)\left[\sum_{j=1}^{n}EX_j^{\,2}I\left(X_j\leqslant N\right)+N^2P\left(X_j>N\right)\right]^{v/2}$$

$$\leqslant C\sum_{n=1}^{\infty}n^{p-1-v}\left(\log n\right)^{v}+C\sum_{n=1}^{\infty}n^{p-2-v/2}\left(\log n\right)^{v}g(n)$$

$$\leqslant\begin{cases}C\sum_{n=1}^{\infty}n^{p-1-v}\left(\log n\right)^{v}+\ C\sum_{n=1}^{\infty}n^{p-2-v/2+\tau}\left(\log n\right)^{v}, & \text{if } A_1 \text{ holds}\\ C\sum_{n=1}^{\infty}n^{p-1-v}\left(\log n\right)^{v}+\ C\sum_{n=1}^{\infty}n^{p-2-v/2+\gamma+1}\left(\log n\right)^{v}\dfrac{g(n)}{nh^{\gamma}(n)}, & \text{if } A_2 \text{ holds}\end{cases}<\infty. \tag{15}$$

For $n>N,\ j\geqslant 1,$ let

$$Z'_{nj}=\left(X_j-N\right)I\left(N<X_j\leqslant n\right)+(n-N)I\left(X_j>n\right).$$

By definition of $Z_j$ and $Z'_{nj}$, we get

$$\sum_{j=1}^{n}EZ_j\leqslant\sum_{j=1}^{n}EX_jI\left(X_j>N\right)<n\varepsilon/8,\quad \sum_{j=1}^{n}EZ'_{nj}\leqslant\sum_{j=1}^{n}EX_jI\left(X_j>N\right)<n\varepsilon/8.$$

For $I_4$, we obtain

$$I_4\leqslant C\sum_{n=1}^{\infty}n^{p-2}P\left(\sum_{j=1}^{n}Z_j+\sum_{j=1}^{n}EZ_j>n\varepsilon/2\right)\leqslant C\sum_{n=1}^{\infty}n^{p-2}P\left(\sum_{j=1}^{n}Z_j>3n\varepsilon/8\right)\leqslant C\sum_{n=1}^{\infty}n^{p-2}P\left(\sum_{j=1}^{n}Z'_{nj}>3n\varepsilon/8\right)+C\sum_{n=1}^{\infty}n^{p-2}P\left(\bigcup_{j=1}^{n}X_j>n\right)$$

$$\leqslant C\sum_{n=1}^{\infty}n^{p-2}P\left(|\sum_{j=1}^{n}(Z'_{nj}-EZ'_{nj})|+\sum_{j=1}^{n}EZ'_{nj}>3n\varepsilon/8\right)+C\sum_{n=1}^{\infty}n^{p-1}P\left(X>n\right)$$

$$\leqslant C\sum_{n=1}^{\infty}n^{p-2}P\left(|\sum_{j=1}^{n}(Z'_{nj}-EZ'_{nj})|>n\varepsilon/4\right)+C\sum_{n=1}^{\infty}n^{p-1}P\left(X>n\right)=:I_{41}+I_{42}. \tag{16}$$

For $I_{42}$, by $E|X|^p<\infty,\ p>1,$ we have

$$I_{42}\leqslant C\sum_{n=1}^{\infty}n^{p-1}\sum_{k=n}^{\infty}P\left(k<X\leqslant k+1\right)\leqslant C\sum_{k=1}^{\infty}P\left(k<X\leqslant k+1\right)\sum_{n=1}^{k}n^{p-1}\leqslant C\sum_{k=1}^{\infty}k^pP\left(k<X\leqslant k+1\right)\leqslant CEX^p<\infty. \tag{17}$$

For $I_{41}$, taking $v>\max\{2,\ p\}$, we have

$$I_{41}\leqslant C\sum_{n=1}^{\infty}n^{p-2-v}E|\sum_{j=1}^{n}(Z'_{nj}-EZ'_{nj})|^v\leqslant C\sum_{n=1}^{\infty}n^{p-2-v}\left\{\sum_{j=1}^{n}E|Z'_{nj}|^v+g\left(n\right)\left(\sum_{j=1}^{n}E|Z'_{nj}|^2\right)^{v/2}\right\}=:I_{411}+I_{412}.$$

By Lemma 4 and $I_{42}<\infty$, we get

$$I_{411}\leqslant C\sum_{n=1}^{\infty}n^{p-2-v}\sum_{j=1}^{n}\left[EX_j^{\,v}I\left(X_j\leqslant n\right)+n^vP\left(X_j>n\right)\right]\leqslant C\sum_{n=1}^{\infty}n^{p-1-v}\left[EX^vI\left(X\leqslant n\right)+n^vP\left(X>n\right)\right]$$

$$\leqslant C\sum_{n=1}^{\infty}n^{p-1-v}EX^vI\left(X\leqslant n\right)+C\sum_{n=1}^{\infty}n^{p-1}P\left(X>n\right)\leqslant C\sum_{n=1}^{\infty}n^{p-1-v}\sum_{k=1}^{n}EX^vI\left(k-1<X\leqslant k\right)+CEX^p$$

$$\leqslant C\sum_{k=1}^{\infty}k^{p-v}EX^vI\left(k-1<X\leqslant k\right)+CEX^p\leqslant CEX^p<\infty. \tag{18}$$

For $I_{412}$, we have

$$I_{412}=C\sum_{n=1}^{\infty}n^{p-2-v}g\left(n\right)\left\{\sum_{j=1}^{n}[EX_j^{\,2}I(X_j\leqslant n)+n^2P(X_j>n)]\right\}^{v/2}\leqslant C\sum_{n=1}^{\infty}n^{p-2-v/2}g\left(n\right)\left[EX^2I\left(X\leqslant n\right)+n^2P\left(X>n\right)\right]^{v/2}.$$

The proof of $I_{412}<\infty$ will be conducted under the following two cases.

**Case 1:** When $p\geqslant 2$, the $E|X|^p<\infty$ implies that $EX^2<\infty$, taking $v>\max\left\{p,\ 2(p-1+\tau),\ 2(p+\gamma)\right\}$, then

$$I_{412}\leqslant C\sum_{n=1}^{\infty}n^{p-2-v/2}g\left(n\right)\left(EX^2\right)^{v/2}\leqslant\begin{cases}C\sum_{n=1}^{\infty}n^{p-2-v/2+\tau}, & \text{if } A_1 \text{ holds}\\ C\sum_{n=1}^{\infty}n^{p-2-v/2+1+\gamma}\dfrac{g(n)}{nh^{\gamma}(n)}, & \text{if } A_2 \text{ holds}\end{cases}<\infty. \tag{19}$$

**Case 2:** When $1<p<2$, under the condition $E|X|^p<\infty$, taking $v>\max\left\{2,\ \frac{2(p-1+\tau)}{p-1},\ \frac{2(p+\gamma)}{p-1}\right\}$, we have

$$I_{412}\leqslant C\sum_{n=1}^{\infty}n^{p-2-v/2}g(n)\left[EX^p n^{2-p}I(X\leqslant n)+EX^p n^{2-p}P(X>n)\right]^{v/2}\leqslant C\sum_{n=1}^{\infty}n^{p-2-v/2}n^{(2-p)v/2}g(n)\left(EX^p\right)^{v/2}$$

$$\leqslant C\sum_{n=1}^{\infty}n^{p-2+v/2-pv/2}g(n)\leqslant\begin{cases}C\sum_{n=1}^{\infty}n^{p-2+v/2-pv/2+\tau}, & \text{if } A_1 \text{ holds}\\ C\sum_{n=1}^{\infty}n^{p-2+v/2-pv/2+1+\gamma}\frac{g(n)}{nh^{\gamma}(n)}, & \text{if } A_2 \text{ holds}\end{cases}<\infty. \tag{20}$$

From (14)-(20), the proof of Theorem 3 is completed.

**Proof of Theorem 4** We get $E|X|<\infty$ by $E|X|h^{\delta}(|X|)<\infty$. Consequently, there exists a positive integer $A$ such that $EXI(X>A)<\varepsilon/8$. For $j\geqslant 1$, let

$$Y_j=X_jI(X_j\leqslant A)+AI(X_j>A),\ Z_j=X_j-Y_j=(X_j-A)I(X_j>A),$$

then

$$\sum_{n=1}^{\infty}n^{-1}P\left(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}(X_j-EX_j)|>\varepsilon n\right)\leqslant\sum_{n=1}^{\infty}n^{-1}P\left(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}(Y_j-EY_j)|>\varepsilon n/2\right)+\sum_{n=1}^{\infty}n^{-1}P\left(\max_{1\leqslant k\leqslant n}|\sum_{j=1}^{k}(Z_j-EZ_j)|>\varepsilon n/2\right)=:I_5+I_6. \tag{21}$$

For $I_5$, be the same as $I_3$ in Theorem 3, taking $v>2(\gamma+1)$, we have

$$I_5\leqslant C\sum_{n=1}^{\infty}n^{-v}\left(\log n\right)^v+C\sum_{n=1}^{\infty}n^{-1-v/2+\gamma+1}\left(\log n\right)^v\frac{g(n)}{nh^{\gamma}(n)}<\infty. \tag{22}$$

Noting $\frac{n}{h^{\delta}(n)}\uparrow$, taking $\frac{n}{h^{\delta}(n)}>A$. For $1\leqslant j\leqslant n$, let

$$Z'_{nj}=\left(X_j-A\right)I\left(A<X_j\leqslant\frac{n}{h^{\delta}(n)}\right)+\left(\frac{n}{h^{\delta}(n)}-A\right)I\left(X_j>\frac{n}{h^{\delta}(n)}\right).$$

By the definitions of $Z_j$ and $Z'_{nj}$, we have

$$\sum_{j=1}^{n}EZ_j<n\varepsilon/8,\ \sum_{j=1}^{n}EZ'_{nj}<n\varepsilon/8.$$

For $I_6$, as $I_4$ in Theorem 3, we have

$$I_6\leqslant C\sum_{n=1}^{\infty}n^{-1}P\left(|\sum_{j=1}^{n}(Z'_{nj}-EZ'_{nj})|>n\varepsilon/4\right)+C\sum_{n=1}^{\infty}P\left(X>n/h^{\delta}(n)\right)=:I_{61}+I_{62}. \tag{23}$$

By (11), we have $\sup\frac{h(x)}{h(x/h(x))}\leqslant C$. Since $0<h(x)\uparrow,\ 0<\delta\leqslant 1,$ we get

$$I_{62}\leqslant C\sum_{n=1}^{\infty}P\left\{Xh^{\delta}(X)>\frac{n}{h^{\delta}(n)}h^{\delta}\left(\frac{n}{h^{\delta}(n)}\right)\right\}\leqslant C\sum_{n=1}^{\infty}P\left\{Xh^{\delta}(X)>n\left[\frac{h\left(\frac{n}{h(n)}\right)}{h(n)}\right]^{\delta}\right\}\leqslant C\sum_{n=1}^{\infty}P\left\{Xh^{\delta}(X)>\frac{n}{C}\right\}$$

$$\leqslant C\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}P\left(k<CXh^{\delta}(X)\leqslant k+1\right)\leqslant C\sum_{k=1}^{\infty}kP\left(k<CXh^{\delta}(X)\leqslant k+1\right)\leqslant CEXh^{\delta}(X)<\infty. \tag{24}$$

For $I_{61}$, taking $v>2,$ we have

$$I_{61}\leqslant C\sum_{n=1}^{\infty}n^{-1-v}E|\sum_{j=1}^{n}(Z'_{nj}-EZ'_{nj})|^v\leqslant C\sum_{n=1}^{\infty}n^{-1-v}\left\{\sum_{j=1}^{n}E|Z'_{nj}|^v+g(n)\left(\sum_{j=1}^{n}E|Z'_{nj}|^2\right)^{v/2}\right\}=:I_{611}+I_{612}. \tag{25}$$

By $h(x)\uparrow,\frac{x}{h^{\delta}(x)}\uparrow$ and $I_{62}<\infty$, we have

$$I_{611}=C\sum_{n=1}^{\infty}n^{-1-v}\sum_{j=1}^{n}\left[EX_j^{\,v}I\left(X_j\leqslant\frac{n}{h^{\delta}(n)}\right)+\left(\frac{n}{h^{\delta}(n)}\right)^{v}P\left(X_j>\frac{n}{h^{\delta}(n)}\right)\right]$$
$$\leqslant C\sum_{n=1}^{\infty}n^{-v}\left[EX^{v}I\left(X\leqslant\frac{n}{h^{\delta}(n)}\right)+\left(\frac{n}{h^{\delta}(n)}\right)^{v}P\left(X>\frac{n}{h^{\delta}(n)}\right)\right]\leqslant C\sum_{n=1}^{\infty}n^{-v}EX^{v}I\left(X\leqslant\frac{n}{h^{\delta}(n)}\right)+C\sum_{n=1}^{\infty}P\left(X>\frac{n}{h^{\delta}(n)}\right)$$
$$\leqslant C\sum_{n=1}^{\infty}n^{-v}\sum_{k=1}^{n}EX^{v}I\left(\frac{k-1}{h^{\delta}(k-1)}<X\leqslant\frac{k}{h^{\delta}(k)}\right)+CEXh^{\delta}(X)\leqslant C\sum_{k=1}^{\infty}EX^{v}I\left(\frac{k-1}{h^{\delta}(k-1)}<X\leqslant\frac{k}{h^{\delta}(k)}\right)\sum_{n=k}^{\infty}n^{-v}+CEXh^{\delta}(X)$$
$$\leqslant C\sum_{k=1}^{\infty}k^{-v+1}EX^{v}I(\frac{k-1}{h^{\delta}(k-1)}<X\leqslant\frac{k}{h^{\delta}(k)})+CEXh^{\delta}(X)\leqslant C\sum_{k=1}^{\infty}EXI\left(\frac{k-1}{h^{\delta}(k-1)}<X\leqslant\frac{k}{h^{\delta}(k)}\right)\frac{1}{h^{\delta(v-1)}(k)}+CEXh^{\delta}(X)$$
$$\leqslant CEX+CEXh^{\delta}(X)<\infty. \tag{26}$$

For $I_{612}$, we have

$$I_{612}=C\sum_{n=1}^{\infty}n^{-1-v}g(n)\left\{\sum_{j=1}^{n}\left[EX_j^{\,2}I\left(X_j\leqslant\frac{n}{h^{\delta}(n)}\right)+\left(\frac{n}{h^{\delta}(n)}\right)^{2}P\left(X_j>\frac{n}{h^{\delta}(n)}\right)\right]\right\}^{v/2}$$
$$\leqslant C\sum_{n=1}^{\infty}n^{-1-v/2}g(n)\left[EX\frac{n}{h^{\delta}(n)}I\left(X\leqslant\frac{n}{h^{\delta}(n)}\right)+EX\frac{n}{h^{\delta}(n)}I\left(X>\frac{n}{h^{\delta}(n)}\right)\right]^{v/2}$$
$$\leqslant C\sum_{n=1}^{\infty}n^{-1-v/2}g(n)\frac{n^{v/2}}{h^{v\delta/2}(n)}[EX]^{v/2}\leqslant C\sum_{n=1}^{\infty}\frac{g(n)}{nh^{v\delta/2}(n)}<\infty. \tag{27}$$

From (21)-(27), the proof of Theorem 4 is completed.

## References


[1] Wang K Y, Wang Y B, Gao Q W. Uniform asymptotics for the finite-time ruin probability of a dependent risk model with a constant interest rate[J]. *Methodology and Computing in Applied Probability*, 2013, **15**(1): 109-124.

[2] Ebrahimi N, Ghosh M. Multivariate negative dependence[J]. *Communications in Statistics-Theory and Methods*, 1981, **10**(4): 307-337.

[3] Liu L. Precise large deviations for dependent random variables with heavy tails[J]. *Statistics & Probability Letters*, 2009, **79**(9): 1290-1298.

[4] Wang K Y, Yang Y, Lin J G. Precise large deviations for widely orthant dependent random variables with dominatedly varying tails[J]. *Frontiers of Mathematics in China*, 2012, **7**(5): 919-932.

[5] Qiu D H, Chen P Y. Complete and complete moment convergence for weighted sums of widely orthant dependent random variables[J]. *Acta Mathematica Sinica*, *English Series*, 2014, **30**(9): 1539-1548.

[6] Liu X, Shen Y, Yang J, *et al*. Complete moment convergence of widely orthant dependent random variables[J]. *Communications in Statistics-Theory and Methods*, 2017, **46**(14): 7256-7265.

[7] Wang Y B, Cui Z L, Wang K Y, *et al*. Uniform asymptotics of the finite-time ruin probability for all times[J]. *Journal of Mathematical Analysis and Applications*, 2012, **390**(1): 208-223.

[8] Chen P Y, Wang L, Wang Y B. Uniform asymptotics for the finite-time ruin probabilities of two kinds of nonstandard bidimensional risk models[J]. *Journal of Mathematical Analysis and Applications*, 2013, **401**(1): 114-129.

[9] Shen A T. Bernstein-type inequality for widely dependent sequence and its application to nonparametric regression models [J]. *Abstract and Applied Analysis*, 2013, **2013**(1): 862602.

[10] Wang X J, Xu C, Hu T C, *et al*. On complete convergence for widely orthant dependent random variables and its applications in nonparametric regression models[J]. *TEST*, 2014, **23**(3): 607-629.

[11] Ding Y, Wu Y, Wang X J, *et al*. Complete convergence for weighted sums of widely orthant dependent random variables[J]. *Applied Mathematics—A Journal of Chinese Universities Series A*, 2015, **30**(4): 417-424(Ch).

[12] Song M Z, Chu Y. Convergence of moving average processes for WOD random variable[J]. *J Wuhan Univ* (*Nat Sci Ed*), 2021, **67**(5): 452-460(Ch).

[13] Song M Z, Wu Y F, Chu Y. Complete $q$th-moment convergence of moving average process for $m$-WOD random variable[J]. *Wuhan University Journal of Natural Sciences*, 2022, **27**(5): 396-404.

[14] Song M Z, Wu Y F. Complete convergence for moving average processes under *m*-WOD random variables[J]. *Wuhan University Journal of Natural Sciences*, 2024, **29**(6): 529-538.

[15] Fang H Y, Ding S S, Li X Q, *et al*. Asymptotic approximations of ratio moments based on dependent sequences[J]. *Mathematics*, 2020, **8**(3): 361.

[16] Spitzer F. A combinatorial lemma and its application to probability theory[J]. *Transactions of the American Mathematical Society*, 1956, **82**(2): 323-339.

[17] Bai P, Chen P Y, Sung S H. On complete convergence and the strong law of large numbers for pairwise independent random variables[J]. *Acta Mathematica Hungarica*, 2014, **142**(2): 502-518.

[18] Chen P Y, Hak S. A Spitzer-type law of large numbers for widely orthant dependent random variables[J]. *Statistics & Probability Letters*, 2019, **154**: 108544.

[19] Chen P Y, Sung S H. Complete convergence for weighted sums of widely orthant-dependent random variables[J]. *Journal of Inequalities and Applications*, 2021, **2021**(1): 45.

[20] Wu Q Y. *Probability Limit Theory for Mixing Sequences* [M]. Beijing: Sicence Press of China, 2006(Ch).

## ***m***-宽相依随机变量序列最大部分和的完全收敛性

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摘要：本文在一般条件下获得了*m*-宽相依随机变量序列最大部分和的完全收敛性，为*m*-WOD随机变量建立了更广泛的完全收敛定理。*m*-宽相依随机变量序列包含多种类型的相依随机变量序列，因此，本文结论改进并推广了现有文献中的相关结果。

关键词：*m*-WOD随机变量序列；完全收敛性；Spitzer’s大数定律

□