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Optimal Investment Strategies of Target Benefit Pension Plans Under Linear Rational Wishart Mortality Risk

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Abstract: In this paper, we investigate the optimal investment strategy for a target-return pension plan under a Wishart model using stochastic optimal control methods. We consider the fact that mortality and interest rates are affected by a combination of systematic and idiosyncratic factors, which are modelled by a Wishart model with positive values. On this basis, we analyse the correlation between stochastic mortality and stochastic wage rates for plan members, as well as wage movements and financial market volatility. It is shown that financial market parameters and wage rates have a significant impact on the optimal investment strategy. We provide the optimal investment strategy and the optimal yield adjustment strategy for target-return pensions. This study provides a valuable reference for policy formulation and implementation of target-return pension plans in China.

Key words: target benefit pension plans; Wishart mortality risk; intergenerational risk; stochastic optimal control

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0 Introduction

In the current global context of accelerated population aging and rising medical costs for retirees, ensuring the adequacy and stability of pension funds has become crucial. As a result, issues related to optimal investment strategies and structural design of pension plans have increasingly garnered attention in recent years. Traditionally, pension plans are mainly divided into two types: Defined Benefit (DB) and Defined Contribution (DC). In DB plans, future pension benefits are predetermined, while contribution rates are adjusted as needed, with fund managers bearing the investment risks. In DC plans, contribution rates are predetermined, and pension benefits depend on investment performance, with participants bearing the investment risks. Due to factors such as population aging and increased lifespans, it is becoming increasingly unreasonable for fund managers or participants to bear all the risks alone. Therefore, reforming existing pension plans is necessary. To this end, some scholars have proposed hybrid pension plans that combine the features of DB and DC, such as Canada's Target Benefit Plan (TBP) and Japan's risk-sharing plan. More and more Canadian fund managers are choosing TBP and other hybrid pension plans to replace traditional DB and DC plans. The characteristic of TBP is fixed contributions and preset target benefit levels, with risks shared among members of different generations. This plan pro-

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motes risk sharing among members and helps maintain relatively stable benefit levels over the long term.

The core characteristic of Target Benefit Plans (TBPs) lies in their risk-sharing mechanism, where all risks are borne collectively by the plan members. This risk sharing is not carried solely by individual members but is distributed among different generations. Due to the lack of external guarantees, the plan encourages risk transfer among members, allowing for temporary subsidies between generations, including future members who have not yet joined the plan, to help maintain relatively stable benefit levels. Research by Gollier and Cui *et al*[1-2] indicated that compared to traditional DB or DC plans, plans with effective intergenerational risk sharing can enhance overall welfare. This means that by spreading risks across generations, TBPs can provide more stable retirement income, reducing uncertainty and potential losses caused by market fluctuations or extended longevity.

In life insurance and annuity products, interest rate risk and mortality risk are the two main risk factors. Traditionally, actuaries have considered mortality risk to be more difficult to model than interest rate risk. According to the law of large numbers, by holding a sufficiently large portfolio of similar contracts, mortality risk can be diversified. Therefore, actuaries often use deterministic methods to model mortality rates while assuming that interest rates are random over the long term.

When stochastic mortality models emerged in the 1990s, the actuarial community assumed that mortality risk was independent of interest rate risk. This assumption seemed reasonable in the short term. However, catastrophic risks such as earthquakes or severe pandemics could affect the economy and financial markets in the short term, suggesting that we should consider the potential dependence between mortality and interest rates. In the long run, it is intuitive that demographic changes will affect the economy. For example, research by Favero *et al*[3] suggested that slowly evolving averages are likely related to population trends. Maurer[4] explored how demographic shifts impact the value of financial assets and proposed a continuous-time overlapping generations model where birth and death rates are stochastic. His model indicated that demographic transitions can explain most of the time variation in real interest rates, stock premiums, and conditional stock price volatility. Dacorogna and Cadena[5] provided empirical evidence showing that economic and financial market behavior changed during periods with higher mortality. Nicolini[6] also indicated that the increase in adult life expectancy during the 17th and 18th centuries might be a key factor explaining asset accumulation and interest rate declines in pre-industrial England. In summary, while the independence assumption may be a useful simplification in the physical world, it may not be entirely applicable in the pricing world. Therefore, a pricing framework allowing for dependency between mortality and interest rates, as suggested by Miltersen and Liu *et al*[7-8], is more reasonable.

This paper establishes a joint mortality and interest rate risk model, where the price of the risk-free asset is based on the Wishart process. Due to its analytical properties, this allows for considerable general correlation between mortality and interest rate risks. In the context of the current social security system, Wang *et al*[9] solved the continuous-time stochastic optimal control problem for TBPs models and obtained closed-form expressions for optimal asset allocation as well as benefit adjustment strategies that combine benefit risk with intergenerational transfers. However, their optimal investment in high-risk assets is a fixed proportion relative to various predetermined targets concerning future potential shortfalls. Therefore, building upon Wang *et al*'s research, this study further considers the impact of interest rate risk under stochastic mortality using a new model proposed by Kaas *et al*[10], the linear-rational Wishart model, which allows for joint modeling of mortality and interest rate risks.

In our model, the pension fund can invest in both risk-free and risky assets, with its income expenditures depending on the fund's wealth and specific income targets. The plan trustees aim to minimize the cumulative squared sum of linear deviations between income expenditures and preset targets over the entire distribution period, while minimizing the portfolio risk at the end of the period. Yong *et al*[11] provided a detailed description of this linear quadratic optimal control problem. Vigna[12] considered both the investment risks faced by individuals and the annualized risks, using a series of medium-term goals and retirement goals linked to the expected net replacement rate in a discrete-time model. According to James ' overlapping generations (OLG) model[13], plan members are divided into young and old periods. Mortality and interest rate joint risks, as important background risks, have a significant impact on the actual value of target benefit pensions. Zhang *et al*[14] investigated the impact of attendant mortality and intergenerational risk on the most investment strategies of target return pensions

under inflation risk, based on which the impact of mortality on interest rates is considered to change the price of risk-free assets. First, assuming mortality follows a Wishart process, we present the price of risk-free assets under the combined influence of mortality and interest rates. Second, we extend the existing optimal investment strategy models to a stochastic interest rate risk environment. Finally, we solve the Hamilton-Jacobi-Bellman (HJB) equation of the model to obtain the optimal investment strategy and optimal income adjustment strategy for target benefit pensions under the joint environment of mortality and interest rates based on the Wishart process.

The structure of this paper is as follows. Section 1 describes the modeling framework. Section 2 covers the main properties of the optimal investment strategy for target benefit pensions under a linear-rational Wishart model, while Section 3 implements the model through some numerical examples. Section 4 summarizes the paper.

1 Model Formulation

This article discusses a collective pension scheme that involves continuous new participants and retirees at various points in time. Its purpose is to achieve the expected return level set by the initiator, which increases at a predetermined rate over time. At any given moment, the actual pension received by retired members is determined by an external salary process, the randomness of which is considered to be associated with fluctuations in the financial market. The pension fund mainly invests in two types of representative assets: risk-free assets and risky assets (stocks). The plan is managed by a trustee who is responsible for adjusting the asset portfolio and setting the pension payments to retirees to keep the payment level as consistent as possible with the target return while ensuring that it does not overburden future groups (or leave them with too much capital). This issue can be viewed as a stochastic control problem.

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\mathcal{F}=\{\mathcal{F}_t\}_{t\geqslant 0}$, where $\mathcal{F}_t$ is the augmentation of the natural filtration generated by a two-dimensional standard Brownian motion. This filtration is assumed to be complete and right-continuous, encompassing both financial market information and salary information at the time of retirement for the members. It is suitable for the initiators of the pension plan, and P is a real-world probability measure on this space.

1.1 Assessment of Direct Reputation

We assume that the process $(\boldsymbol{Z}_t)_{t\geqslant 0}$ is a d-dimensional Wishart process. Given a $d\times d$ matrix-valued Brownian motion $\boldsymbol{W}_1$, the Wishart process without jumps Z_t is defined as the solution to the following $d\times d$-dimensional stochastic differential equation

$$\mathrm{d}\boldsymbol{Z}_t=(\kappa\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q}+\boldsymbol{H}\boldsymbol{Z}_t+\boldsymbol{Z}_t\boldsymbol{H}^{\mathrm{T}})\mathrm{d}t+\sqrt{\boldsymbol{Z}_t}\,\mathrm{d}\boldsymbol{W}_1(t)\boldsymbol{Q}+\boldsymbol{Q}^{\mathrm{T}}\mathrm{d}\boldsymbol{W}_1^{\mathrm{T}}(t)\sqrt{\boldsymbol{Z}_t},\ t\geqslant 0. \tag{1}$$

Here, Z_0 is the initial condition, $\boldsymbol{W}_1$ is the matrix-valued Brownian motion. $\boldsymbol{H}\in\boldsymbol{M}_d$ (the set of real $d\times d$ matrices) $\boldsymbol{Q}\in\boldsymbol{GL}_d$ (the set of invertible real $d\times d$ matrices) and $\boldsymbol{Q}^{\mathrm{T}}$ its transpose. Bru[15] proved existence and uniqueness of a weak solution for equation (1).

1.2 Financial Market

Assume that the financial market offers two basic assets for the pension trustee to choose from a risk-free asset (bank account) and a risky asset (stock).

Proposition 1 The value of the risk-free asset $S_0(t)$ changes over time according to

$$\mathrm{d}S_0(t)=r_tS_0(t)\mathrm{d}t,\ t\geqslant 0. \tag{2}$$

where r_t represents the interest rate influenced by the Wishart mortality effect. According to research by Deelstra *et al*[16], the interest rate process is affected by the mortality rate process. Provide the interest rate process $r_t=r_0+\mathrm{Tr}(\boldsymbol{R}\boldsymbol{Z}_t)$ and the mortality rate process $\gamma_t=\gamma_0+\mathrm{Tr}(\boldsymbol{M}\boldsymbol{Z}_t)$. Here, $\boldsymbol{R}$ is the interest rate weight, and $\boldsymbol{M}$ is the mortality rate weight.

At time t, the price of the stock (risky asset) is defined as $S_1(t)$, and the value of the risky asset is described by a stochastic differential equation

$$\mathrm{d}S_1(t)=S_1(t)\left[\mu\mathrm{d}t+\sigma\mathrm{d}W_2(t)\right],\ t\geqslant 0. \tag{3}$$

Here, μ is the appreciation rate of the stock, σ is the volatility, μ and σ both and are positive constants. $W_2(t)$ is a standard Brownian motion. To preclude arbitrage opportunities, it is assumed that $\mu>r_0$.

1.3 Members and Plan Provisions

Consider a plan consisting of active members and retired members. When active members contribute to the pension fund, retired members receive benefits from the pension fund. Based on James' OLG model[13], assume the number of active members is n, and the number of retired members is $\frac{n}{\theta+1}$. All members join the TBP plan at age a and retire at age r, subject to $s(a)=1$ and $a\leqslant x\leqslant \omega$, the survival function

$$s(x)=\mathrm{e}^{-A(x-a)-\frac{B}{\ln c}\left(c^{x}-c^{a}\right)}. \tag{4}$$

where A is the baseline mortality coefficient, B is the age-sensitive mortality parameter, and c is the exponential scaling factor for survival probability. Assuming the annual salary of retirees can be expressed as

$$\mathrm{d}S(t)=S(t)\left(\alpha\mathrm{d}t+\eta\mathrm{d}\bar{W}(t)\right),\ t\geqslant 0. \tag{5}$$

where $\alpha\in\mathbb{R}^{+}$ is the instantaneous growth rate of wage expectations, $\eta\in\mathbb{R}$ is the instantaneous volatility of wages, and $\bar{W}$ is a standard Brownian motion, which is assumed to $W(t)$ have a correlation coefficient ρ with under the P measure .

The pension plan under discussion provides members with a lifetime annuity starting from their retirement age r. The initial annual pension payment rate is assumed to be a portion of the final salary $S(t)$ at retirement. Specifically, for retired members with a final salary, the initial pension payment rate is assumed to be $f(t)S(t)$, where $f(t)$ can be considered as the immediate replacement rate applicable to new retirees. To determine the annual pension payment rate for a member retiring at age x at time t, a new variable $L(x,t)$ is introduced, which is the assumed salary of the member at the time of retirement ($x-r$ years prior to retirement). This variable is defined as

$$L(x,t)=S(t)\mathrm{e}^{-\alpha(x-r)},\ t\geqslant 0, x\geqslant r. \tag{6}$$

Starting with the salary of a member retiring at time t, predictions are made using an exponential growth rate α. When $\eta>0$ and $x>r$, this is clearly different from the actual salary at retirement, as well as the expected difference between the actual salary and the assumed salary increasing with age. However, due to the decreasing number of older members, the overall impact of this assumption on the results for reasonable values of η is negligible. Additionally, this simplified assumption provides a key advantage as it allows for explicit resolution of the optimal control problem presented in the next section.

Assuming the plan adjusts the cost of living for pensions at a fixed annual percentage of ζ, then for a retiree aged x at time t (retiring $x-r$ years early), there are two adjustment factors acting on the wage rate $L(x,t)$: $\mathrm{e}^{\zeta(x-r)}$ represents the cost-of-living adjustment factor; $f(t)$ represents the control variable applicable to all retirees at time t. The annual pension payment rate for retirees aged x at time t is denoted as $B(x,t)$.

$$B(x,t)=f(t)L(x,t)\mathrm{e}^{\zeta(x-r)}=f(t)S(t)\mathrm{e}^{-(\alpha-\zeta)(x-r)},\ x\geqslant r. \tag{7}$$

This process includes the automatic increase of the cost of living and adjustments to the burden capacity driven by emerging experiences.

Assuming the plan has a pre-set target total retirement benefit B^{*} at time 0, representing the initial payment rate provided by the plan to the current retiree population. The benefit target grows exponentially at a rate of β, but it does not necessarily equate to the automatic cost-of-living adjustments applicable ζ for pension payments. Then at time $t(>0)$, the total target benefit is $B^{*}\mathrm{e}^{\beta t}$.

The actual total payment rate for all retirees at time t is $B(t)$, that is

$$\begin{aligned}B(t)&=\frac{n}{\theta+1}s(x)B(x,t)\\&=\frac{n}{\theta+1}s(x)f(t)S(t)\mathrm{e}^{-(\alpha-\zeta)(x-r)}.\end{aligned} \tag{8}$$

Meanwhile, contributions are injected into the fund. Assume that each active member has an instantaneous contribution rate of c_0 at time 0, expressed in terms of \$1 per year. It is assumed that contributions increase over time, with a growth rate identical to the deterministic component of the growth rate α applicable to wages. This assumption is motivated by three factors. First, although salaries may experience random fluctuations (shocks), these fluctuations are not substantial in practice, making it reasonable to ignore them for the purpose of calculating contributions. Second, this article aims to focus more on return risk rather than contribution risk. Third, simplifying the model in this way allows for an explicit solution to the optimal control problem. Therefore, the total instantaneous contribution rate of all active members at time t is

$$C(t)=ns(x)c_0\mathrm{e}^{\alpha t},\ t\geqslant 0. \tag{9}$$

In the following sections, based on the dynamics of fund investments, contributions from active members, and benefits paid to all retirees, the wealth process of the pension fund for the aforementioned pension scheme will be constructed. Building upon the objectives discussed previously, a continuous-time stochastic optimal control problem for the target benefit plan is established.

1.4 Wealth Process Under the Wishart Mortality Process

Proposition 2 Assume that the pension plan managers can allocate funds into two types of assets mentioned in (2) and (3): risk-free assets and high-risk assets. They use the returns from these investments to pay for retirement benefits. Let $\pi(t)$ denote the proportion of funds invested in risk assets at time t, then the entire wealth variation process can be described by a stochastic differential equation.

$$\mathrm{d}X(t)=X(t)\left[\pi\frac{\mathrm{d}S_1(t)}{S_1(t)}+(1-\pi)\frac{\mathrm{d}S_0(t)}{S_0(t)}\right]+\left[C(t)-B(t)\right]\mathrm{d}t, \tag{10}$$

The value of the wealth process under intergenerational risk follows the dynamic process below:

$$\begin{cases}\mathrm{d}X(t)=X(t)\left\{\left[\pi\mu+r_0+\mathrm{Tr}\left(\boldsymbol{RZ}_t\right)-\pi r_0-\pi\mathrm{Tr}\left(\boldsymbol{RZ}_t\right)\right]\mathrm{d}t+\pi\sigma\mathrm{d}W_2(t)\right\}+\left[C(t)-B(t)\right]\mathrm{d}t,\\ X(0)=x_0.\end{cases} \tag{11}$$

Proof

$$\begin{aligned}\mathrm{d}X(t)&=X(t)\left[\pi\frac{\mathrm{d}S_1(t)}{S_1(t)}+(1-\pi)\frac{\mathrm{d}S_0(t)}{S_0(t)}\right]+\left[C(t)-B(t)\right]\mathrm{d}t\\&=X(t)\left\{\pi\left[\mu\mathrm{d}t+\sigma\mathrm{d}W_2(t)\right]+(1-\pi)r_t\mathrm{d}t\right\}+\left[C(t)-B(t)\right]\mathrm{d}t\\&=X(t)\left\{\pi\left[\mu\mathrm{d}t+\sigma\mathrm{d}W_2(t)\right]+(1-\pi)\left[r_0+\mathrm{Tr}\left(\boldsymbol{RZ}_t\right)\right]\mathrm{d}t\right\}+\left[C(t)-B(t)\right]\mathrm{d}t\\&=X(t)\left\{\left[\pi\mu+r_0+\mathrm{Tr}\left(\boldsymbol{RZ}_t\right)-\pi r_0-\pi\mathrm{Tr}\left(\boldsymbol{RZ}_t\right)\right]\mathrm{d}t+\pi\sigma\mathrm{d}W_2(t)\right\}+\left[C(t)-B(t)\right]\mathrm{d}t.\end{aligned} \tag{12}$$

The plan faces discontinuity risk (due to intergenerational transfers or excessively high or low funding) that can be measured by the squared deviation between the actual final value of the fund $X(t)$ and the predetermined final target. The earnings risk is assessed by considering the difference between the actual total earnings expenditure and the updated total target earnings $B^*\mathrm{e}^{\beta t}$. The trustees of the pension plan aim to minimize these two types of risks simultaneously. In the continuous-time framework, the goal of asset allocation and earnings distribution is to minimize the expected discounted loss over the remaining time, where "loss" refers to the defined earnings risk and discontinuity risk. Considering that participants may be concerned about both the risk of insufficient earnings (i.e., earnings below the target level) and the volatility of earnings (i.e., deviation from the target level), it is reasonable to include linear and quadratic terms of earnings risk in the loss function. Let the objective function at time t be $J(t,x,s)$, with fund value x and hourly wage s, defined as:

$$\begin{cases}J(t,x,s)=E_{\pi,f}\left\{\int_t^T\left[\left(B(u)-B^*\mathrm{e}^{\beta t}\right)^2-\lambda_1\left(B(u)-B^*\mathrm{e}^{\beta t}\right)\right]\times\mathrm{e}^{-r_0 s}\mathrm{d}u+\lambda_2\left(X(T)-x_0\mathrm{e}^{r_0T}\right)^2\mathrm{e}^{-r_0T}\right\},\\ J(T,x,s)=\lambda_2\left(X(T)-x_0\mathrm{e}^{r_0T}\right)^2\mathrm{e}^{-r_0T}.\end{cases} \tag{13}$$

In this context, both λ_1 and λ_2 are non-negative constants that represent the penalty weights for negative deviations between $B(u)$ and $B^*\mathrm{e}^{\beta t}$ and the achievement of the final fund objective. The expectation in (13) is conditional on time, value, and hourly wage. The choices of λ_1 and λ_2 reflect the unique risk balance among the stakeholders in each TBP. The specific combination of penalty weights is an integral part of the overall design of the plan and is not within the purview of the plan trustees to alter.

The value function is defined as:

$$\phi(t,x,s):=\min_{(\pi,f)}J(t,x,s),t,x,s>0. \tag{14}$$

Consequently, we have formulated a continuous-time stochastic optimal control framework to model the target benefit pension plan.. In the following sections, our goal is to find the optimal strategy that solves this optimal control problem.

2 Model Solution

In this section, we employ the standard method to solve the optimal control problem (14) and derive a closed-form expression for the optimal policy. The closed-form expression for the optimal policy is denoted as $\left(\pi^*,f^*\right)$. Firstly, we derive HJB equation associated with the stochastic control problem (14). Using variational methods and Ito's formula, we can obtain the following HJB equation:

$$\begin{aligned}\min_{\pi,f}\Big\{&\varphi_t+\left[x\left(r_0+\mathrm{Tr}\left(\boldsymbol{RZ}_t\right)\right)+\pi\mu-\pi r_0-\pi\mathrm{Tr}\left(\boldsymbol{RZ}_t\right)+C(t)-B(t)\right]\varphi_x+\alpha s\varphi_s+\frac{1}{2}\pi^2\sigma^2\varphi_{xx}\\&+\frac{1}{2}\eta^2s^2\varphi_{ss}+\rho\pi\sigma\eta s\varphi_{xs}+\left[\left(B(t)-B^*\mathrm{e}^{\beta t}\right)^2-\lambda_1\left(B(t)-B^*\mathrm{e}^{\beta t}\right)\right]\mathrm{e}^{-r_0t}\Big\}=0,\end{aligned} \tag{15}$$

with boundary conditions

$$\varphi(T,x,s)=\lambda_2\left(x-x_0\mathrm{e}^{r_0T}\right)^2\mathrm{e}^{-r_0T}, \tag{16}$$

where $\varphi_t,\varphi_x,\varphi_{xx},\varphi_s,\varphi_{ss}$ and φ_{xs} are the partial derivatives of $\varphi(t,x,s)$.

The following theorem details the optimal asset allocation and return adjustment policy for the optimal control problem (14). We denote $\delta=\dfrac{\mu-r_t}{\sigma}$ to simplify notation, where δ is the Sharpe ratio of the risky asset.

Theorem 1 Theorem 1 addresses the optimal control problem (15), providing strategies for optimal asset allocation and return adjustments, along with the corresponding expression for the value function.

$$\pi^*(t,x,s)=-\frac{\delta}{\sigma}\left(2x+Q(t)\right), \tag{17}$$

$$f^*(t,x,s)=\frac{\theta+1}{nss(x)}\left[\lambda_2P(t)\left(2x+Q(t)\right)+\lambda_1+2B^*\mathrm{e}^{\beta t}\right], \tag{18}$$

$$\phi(t,x,s)=\lambda_2\mathrm{e}^{-r_0t}P(t)\left(x^2+Q(t)x\right)+K(t), \tag{19}$$

where $P(t)$ and $Q(t)$ are

$$P(t)=\mathrm{e}^{r_0t-2\lambda_2\left(r_0+\mathrm{Tr}\left(\boldsymbol{RZ}_t\right)\right)t}, \tag{20}$$

$$Q(t)=\mathrm{e}^{2t\left(-\mathrm{Tr}\left(\boldsymbol{RZ}_t\right)-2\lambda_2r_0-2\lambda_2\mathrm{Tr}\left(\boldsymbol{RZ}_t\right)\right)}-\frac{2C(t)}{-r_0+\mathrm{Tr}\left(\boldsymbol{RZ}_t\right)+2\lambda_2r_0+2\lambda_2\mathrm{Tr}\left(\boldsymbol{RZ}_t\right)}. \tag{21}$$

Proof First, it is easy to see that the minimization problem (15) with respect to π and f can be accomplished through the following two related minimization problems. The values of π and f are given by the following equations:

$$\min_{\pi}\left\{\varphi_t+\left[xr_t-\pi r_t+\pi\mu+C(t)\right]\varphi_x+\alpha s\varphi_s+\frac{1}{2}\pi^2\sigma^2\varphi_{xx}+\rho\pi\sigma\eta s\varphi_{xs}\right\}=0, \tag{22}$$

$$\min_{f}\left\{-B(t)\varphi_x+\frac{1}{2}\eta^2s^2\varphi_{ss}+\left[\left(B(t)-B^*\mathrm{e}^{\beta t}\right)^2-\lambda_1\left(B(t)-B^*\mathrm{e}^{\beta t}\right)\right]\mathrm{e}^{-r_0t}\right\}=0. \tag{23}$$

Based on the above equation, we can solve for

$$\pi^*(t,x,s)=-\frac{\delta\varphi_x+\rho\eta s\varphi_{ss}}{\sigma\varphi_{xx}}, \tag{24}$$

$$f^*(t,x,s)=\frac{\theta+1}{2nss(x)}\left(\mathrm{e}^{r_0t}\varphi_x+\lambda_1+2B^*\mathrm{e}^{\beta t}\right). \tag{25}$$

Next, we find an explicit expression for $\phi(t,x,s)$. Based on the terminal condition (16), we assume the form of $\phi(t,x,s)$ to be

$$\varphi(t,x,s)=\lambda_2\mathrm{e}^{-r_0t}P(t)\left(x^2+Q(t)x\right)+R(t)xs+U(t)s^2+V(t)s+K(t), \tag{26}$$

where $P(t),Q(t),R(t),U(t),V(t),K(t)$ are functions to be determined. From the boundary condition (16), it can be seen that

$$P(T)=1,\ Q(T)=-2x_0\mathrm{e}^{r_0T},\ K(T)=x_0^2\mathrm{e}^{r_0T},\quad R(T)=U(T)=V(T)=0. \tag{27}$$

Differentiate each variable in equation (26) yields

$$\begin{aligned}\phi_t&=\lambda_2\mathrm{e}^{-r_0t}\left\{-r_0P(t)\left[x^2+Q(t)x\right]+P(t)Q_tx+P_t\left[x^2+Q(t)x\right]\right\}+R_txs+U_ts^2+V_ts+K_t,\\ \phi_s&=R(t)x+2U(t)s+V(t),\\ \phi_x&=\lambda_2\mathrm{e}^{-r_0t}P(t)\left[2x+Q(t)\right]+R(t)s,\\ \phi_{xx}&=2\lambda_2\mathrm{e}^{-r_0t}P(t),\ \phi_{xs}=R(t),\ \phi_{ss}=2U(t).\end{aligned}$$

Substitute the above expression into equations (22) and (23), then substitute the resulting sum into the HJB equation (15), and combine like terms.

$$\begin{aligned}&\lambda_2\mathrm{e}^{-r_0t}\left[\left(-r_0+2\lambda_2r_t\right)P(t)+p_t\right]x^2+\left[U_t+\left(2\alpha+\eta^2\right)U(t)\right]s^2+\lambda_2\mathrm{e}^{-r_0t}\left[\left(-r_0Q(t)+Q_t\right.\right.\\ &\left.+r_tQ(t)+2C(t)\right)P(t)+P_tQ(t)\left.\right]x+\left[V_t+\alpha V(t)\right.\\ &-\left(\delta^2Q(t)+\delta\rho\eta Q(t)-C(t)+\lambda_2P(t)Q(t)+\lambda_1\right.\\ &\left.+2B^*\mathrm{e}^{\beta t}\right)R(t)\left.\right]s+\left[R_t+\left(r_t-2\delta^2+\alpha-2\delta\rho\eta\right.\right.\\ &\left.\left.-2\lambda_2P(t)\right)R(t)\right]xs+K_t+\left[\left(3\lambda_1+B^*\mathrm{e}^{\beta t}\right)B^*\mathrm{e}^{\beta t}-\lambda_1^2\right.\\ &\left.+\lambda_2C(t)P(t)Q(t)\right]\mathrm{e}^{-r_0t}\\ &=0.\end{aligned} \tag{28}$$

When the coefficients of x^2, s^2, xs, x and s as well as the constant term are all zero, the equation holds true. This leads to the following system of equations:

$$\left(-r_0+2\lambda_2r_t\right)P(t)+P_t=0, \tag{29}$$

$$U_t+\left(2\alpha+\eta^2\right)U(t)=0, \tag{30}$$

$$\left(-r_0Q(t)+Q_t+r_tQ(t)+2C(t)\right)P(t)+P_tQ(t)=0, \tag{31}$$

$$V_t+\alpha V(t)-\left(\delta^2Q(t)+\delta\rho\eta Q(t)-C(t)+\lambda_2P(t)Q(t)+\lambda_1+2B^*\mathrm{e}^{\beta t}\right)R(t)=0, \tag{32}$$

$$R_t+\left(r_t-2\delta^2+\alpha-2\delta\rho\eta-2\lambda_2P(t)\right)R(t)=0, \tag{33}$$

$$K_t+\left[\left(3\lambda_1+B^*\mathrm{e}^{\beta t}\right)B^*\mathrm{e}^{\beta t}-\lambda_1^2+\lambda_2C(t)P(t)Q(t)\right]\mathrm{e}^{-r_0t}=0. \tag{34}$$

Solve equations (29), (30), (31), (32), (33), and (34) under the boundary conditions given in equation (27).

$$P(t)=\mathrm{e}^{r_0t-2\lambda_2\left(r_0+\mathrm{Tr}\left(\boldsymbol{RZ}_t\right)\right)t}, \tag{35}$$

$$Q(t)=\mathrm{e}^{2t\left(-\mathrm{Tr}\left(\boldsymbol{RZ}_t\right)-2\lambda_2r_0-2\lambda_2\mathrm{Tr}\left(\boldsymbol{RZ}_t\right)\right)}-\frac{2C(t)}{-r_0+\mathrm{Tr}\left(\boldsymbol{RZ}_t\right)+2\lambda_2r_0+2\lambda_2\mathrm{Tr}\left(\boldsymbol{RZ}_t\right)}, \tag{36}$$

$$U(t)=R(t)=V(t)=0. \tag{37}$$

3 Numerical Analysis

In this section, we will analyze the optimal investment strategy results for a target-benefit pension plan under linear rational Wishart mortality risk through numerical examples. Similar to Refs. [5, 9-10], the parameter values are assumed as shown in Table 1. By varying the parameter values of the model, we use graphs to intuitively present the optimal investment strategies under different risks.

Table 1 Parameter values for numerical simulation

B^*	T	α	β	κ	μ	σ	θ	A	B	η	a
100	20	0.03	0.02	3	0.05	0.15	5.32%	0.000 22	0.000 002 7	0.01	30
r	ζ	c_0	x_0	ρ	λ_1	λ_2	r_0	n	c	s_0	
65	0.02	0.1	4 000	0.1	15	0.2	0.01	100	1.124	1	

This study examines how the dependency structure between mortality rates and interest rates influences investment strategies for target return pension funds. We posit that both mortality and interest rates are shaped by a combination of systematic and idiosyncratic factors. To maintain positivity in modeling, we adopt frameworks such as the multi-factor Cox-Ingersoll-Ross (CIR) process and the Wishart process, consistent with the methodology proposed by Liu *et al*[8]. This modeling choice holds critical relevance for risk management, particularly when addressing latent dependencies between these financial variables. Among the models evaluated, the Wishart process proves most advantageous, as it accommodates intricate dependency structures while ensuring positivity. The analysis herein centers on a Wishart process formulation to explore these dynamics.

$$\boldsymbol{H}=\begin{pmatrix}-0.5 & 0.4\\ 0.007 & -0.008\end{pmatrix},\ \boldsymbol{Q}=\begin{pmatrix}0.06 & \boldsymbol{Q}_{12}\\ \boldsymbol{Q}_{12} & 0.006\end{pmatrix},$$

$$\boldsymbol{Z}_0=\begin{pmatrix}0.01 & \boldsymbol{Z}_0^{12}\\ \boldsymbol{Z}_0^{12} & 0.01\end{pmatrix},\ \boldsymbol{R}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

This analysis explores how variations in Wishart-distributed mortality risks influence the pricing dynamics of risk-free assets. By systematically adjusting parameters within asset model (1), we assess the sensitivity of risk-free asset valuations. Stability assessments specifically target perturbations to the diagonal elements of matrix $\boldsymbol{Q}$ and sub-diagonal components of matrix $\boldsymbol{Z}$, raising critical questions about the robustness of expected rate metrics under parameter shifts. Empirical evidence from tabulated data reveals an inverse relationship: elevated initial correlations between mortality rates and interest rates correspond to diminished risk-free asset prices, see Table 2 and Table 3. Notably, intensified mortality-interest rate correlations exacerbate price volatility in these ostensibly safe instruments. These findings underscore a dual causal pathway—mortality patterns influencing interest rate structures, which subsequently reshape risk-free asset valuations—thereby compelling investors to continually recalibrate their strategic asset allocations. As mortality-driven interest rate fluctuations ripple through markets, portfolio managers must navigate evolving risk profiles, dynamically shifting investment weights between risk-free and risky assets to optimize returns amidst increasing systemic interdependencies.

Table 2 The impact of matrix Z and the interest rate mortality dependence on risk-free asset prices in the Wishart process

$\boldsymbol{Z}_0^{12}$	ρ_0	Price
−0.002	−0.489 493 6	5.210 447 1
−0.001 5	−0.367 120 2	5.204 596 3
0	0	5.187 083 4
0.001 5	0.367 120 1	5.169 630 0
0.002	0.489 493 6	5.163 825 4

Table 3 The impact of matrix Q and the interest rate mortality dependence on risk-free asset prices in the Wishart process

$\boldsymbol{Q}_{12}$	ρ_0	Price
−0.01	−0.294 221 0	6.658 698 2
−0.006	−0.244 746 8	7.090 873 4
0	0	6.935 373 8
0.006	0.244 746 8	6.381 516 7
0.01	0.294 221 0	5.657 111 0

Figure 1 illustrates that the optimal investment ratio $\pi^*(t)$ exhibits an initial progressive increase, approaching a peak near unity before experiencing a gradual decline toward 0.6 by year 18. This trajectory indicates dynamic adjustments in the allocation to risky assets over time. During the scheme's early phase, when pension returns are particularly unfavorable and risk-free interest rates remain susceptible to mortality-induced fluctuations, pension trustees are compelled to pursue aggressive strategies, channeling funds into riskier assets. Following the recovery phase, however, the investment approach transitions systematically, balancing exposure between risky and risk-free assets as the terminal phase approaches. This evolution reflects adaptive risk management responding to both market conditions and temporal constraints inherent in pension funding dynamics.

Figure 2 investigates how interest rate fluctuations impact the optimal allocation proportion of risk-free assets under distinct Wishart-modeled mortality risk scenarios. Variations in the parameters governing asset model (2), specifically r_t, induce corresponding adjustments in risk-free asset valuation. The illustration reveals a clear inverse relationship: rising interest rates consistently reduce investments in risky assets. Notably, when interest rates reach 0.04, temporal dynamics become increasingly influential. As time progresses, pension fund trustees demonstrate heightened risk aversion, progressively shifting toward full allocation to risk-free assets. This pattern suggests that elevated interest rates combined with extended investment horizons fundamentally alter risk tolerance, prompting conservative portfolio strategies that prioritize capital preservation over risk exposure.

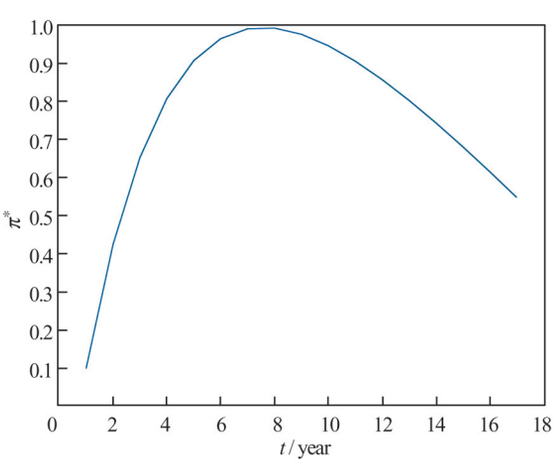

Fig. 1 Evolution of the investment ratio π^* over time t

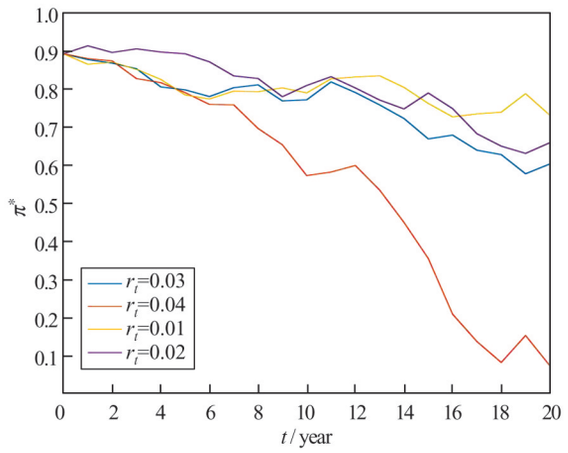

Fig. 2 Variation of the investment ratio π^* over time t at different interest rates r_t

4 Conclusion

This study investigates how mortality-interest rate dependencies influence risk-free asset pricing. We model these dependencies through systematic and idiosyncratic factors constrained to positive values (e. g., multifactor CIR or Wishart frameworks). Our analysis reveals that advanced models capturing nonlinear mortality-interest rate interdependencies—such as the Wishart process—render traditional pairwise linear correlation inadequate for explaining risk-free asset valuations. This underscores critical limitations in conventional risk management when confronting latent dependence structures. Notably, the Wishart model emerges as particularly effective for insurance product pricing due to its capacity to replicate intricate multivariate dependency patterns. These findings advocate for adopting sophisticated stochastic frameworks to address systemic risks arising from unobservable mortality-macroeconomic linkages.

Building on this, this paper explores the optimal investment strategy for a target benefit pension plan that takes mortality risk into account under the Wishart model. We assume that mortality and interest rates are influenced by a combination of systematic and idiosyncratic factors that can be modeled by a positive-valued Wishart model. Based on this model, we obtain the true wealth process of pensions and use the HJB equation to find the optimal asset allocation for a target-return pension plan. By fixing certain parameters, we analyze the effect of these parameters on the optimal asset allocation of the target-return pension plan, thus providing a more

effective investment strategy for investors in target-return pension plans.

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目标收益养老金计划在线性理性Wishart死亡率风险下的最优投资策略

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摘要：本文基于随机最优控制方法，研究了Wishart模型下目标收益养老金计划的最优投资策略。考虑到死亡率和利率受系统性因素与个体特质因素的共同影响，采用取值为正的Wishart模型进行建模。在此基础上，分析了计划成员的随机死亡率与随机工资率之间的相关性，以及工资变动与金融市场波动的关联性。研究表明，金融市场参数和工资率对最优投资策略具有显著影响。本文为目标收益型养老金提供了最优投资策略和最优收益率调整策略，为中国目标收益养老金政策的制定与实施提供了重要参考。

关键词：目标收益养老金计划；Wishart死亡率风险；代际风险；随机最优控制

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